Some links between min-cuts, optimal spanning forests and watersheds

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Abstract Different optimal structures: minimum cuts, minimum spanning forests and shortest-path forests, have been used as the basis for powerful image segmentation procedures. The well-known notion of watershed also falls into this category. In this paper, we present some new results about the links which exist between these different approaches. Especially, we show that min-cuts coincide with watersheds for some particular weight functions.

Keywords: min-cuts, spanning forests, watersheds, shortest-path forests.

Introduction

Min-cuts (graph cuts) and watersheds are two popular tools for image segmentation, which can both be expressed in the framework of graphs and are well suited to computer implementations. Informally, a cut in a graph is a set of edges which, when removed from the graph, separates it into different connected components. Given a set of vertices or subgraphs called markers, the goal of these operators is to find a cut for which each induced component contains exactly one marker, and which best matches a criterion based on the image contents. For example, the criterion is often designed in such a way that the cut is located along the contours of the objects present in the image. To this aim, edges of the pixel adjacency graph can be weighted for example with the inverse of the gradient modulus. The principle of min-cut segmentation is then to find a cut (relative to the markers) which sum of edge weights is minimal [6].

The watershed is a well-known notion from the field of topography, introduced for image segmentation purposes by S. Beucher and C. Lantuéjoul [5]. Intuitively, the watershed of a function (seen as a topographical surface) is composed by the locations from which a drop of water could flow down towards different minima. In a framework of edge-weighted graphs, the watershed is defined in [9, 10] as a cut relative to the regional minima of the weight function, and which satisfies this "drop of water" principle. In [15], Meyer shows the link between minimum spanning forests and flooding

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algorithms, which are most often used to compute watersheds. There is indeed an equivalence between watersheds defined as cuts satisfying the drop of water principle and cuts induced by minimum spanning forests (MinSF) relative to the minima, as proved in [9, 10].

Another point of view on the watershed is studied in [13, 14]. Let us define the "length" of a path as the maximum weight of the edges along this path, then the watershed is defined by these authors as a cut which separates the components of the graph induced by a shortest-path forest rooted in the minima. This definition in terms of shortest-path forest is also the basis for the so-called fuzzy connected image segmentation [2, 16].

The goal of this paper is to clarify the links between these different optimal structures used for image segmentation. To this aim, we first give a set of definitions for these different paradigms in a same unifying framework of edge-weighted graphs. Then, we show that any MinSF is a shortest-path forest, and that the converse is, in general, not true.

At last, we prove a property which links graph cuts and watersheds, through the notion of MinSF. It is well known that the MinSFs, and hence the watersheds, are invariant if an increasing transformation is applied simultaneously to all the weights. For example, if we raise all the weights to a same positive power n, a MinSF remains a MinSF. On the contrary, min-cuts may be different for different values of n. We show that, for any weighted graph, there exists a value n such that min-cuts coincide with cuts induced by maximum spanning forests relative to the markers, furthermore, this will also be true for any number greater than n.

Proofs of the theorems presented in this paper are in [1].

1. Basic notions on graphs

In this section we state basic notions on graphs before presenting the definitions of *extension* and *cut* over a graph, which will be necessary in the sequel of the paper.

We define a graph as a pair $G = (V, E)$ where V is a finite set and E is composed of unordered pairs of elements of V, precisely, E is a subset of $\{\{x, y\} \subseteq V \mid x \neq y\}$. Each element of V is called a *node* or a *vertex (of G)*, and each element of E is called an *edge (of G)*. We denote by $G_\emptyset$ the empty graph, i.e. $G_\emptyset = (\emptyset, \emptyset)$.

Let G be a graph. If $e = \{x, y\}$ is an edge of G, we say that x and y are *adjacent (for G)*. Let $\pi = \langle x_0, \ldots, x_\ell \rangle$ be an ordered sequence of nodes of G, we say that π is a *path from x_0 to x_ℓ in G (or in V)* if for any $i \in [1; \ell]$, x_i is adjacent to x_{i-1}. In this case, we say that x_0 *and x_ℓ are linked for G*. We say that π is a *simple path from x_0 to x_ℓ in G (or in V)* if π is a *path from x_0 to x_ℓ* and if all nodes of π are distinct. Notice that if there exists a path from x_0 to x_ℓ in G, then there exists a simple path from x_0 to x_ℓ. We say that G *is connected* if any two vertices of G are linked for G. Notice that $G_\emptyset$ is connected.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$ then we say that G' *is a subgraph of* G and we write $G' \subseteq G$. Notice that $G_\emptyset$ is a subgraph of any graph.

In the sequel, $G = (V, E)$ will denote a graph.

We say that X *is a connected component of* G if X is a connected subgraph of G which is maximal for this property, i.e. for any connected graph X', $X \subseteq X' \subseteq G$ implies $X' = X$. Notice that $G_\emptyset$ is not a connected component of any non-empty graph, and that $G_\emptyset$ is the connected component of, and only of, $G_\emptyset$.

Let X be a subgraph of G, we denote respectively by $V(X)$ and $E(X)$ the node set and the edge set of X.

Let X and Y be two subgraphs of G. We define $(X \cup Y) = (V(X) \cup V(Y), E(X) \cup E(Y)$ and $(X \cap Y) = (V(X) \cap V(Y), E(X) \cap E(Y))$.

Let X be a subgraph of G. An edge $\{x, y\}$ of G is *adjacent to* X if $\{x, y\} \cap V(X) \neq \emptyset$ and $\{x, y\} \notin E(X)$. In this case, if $x \in V(X)$, either $y \in V(X)$ or y is adjacent to X.

If S is a subset of E, we denote by $\overline{S}$ the complementary set of S in E, that is, $\overline{S} = E \setminus S$.

Let $S \subseteq E$. The *graph induced by* S is the graph whose edge set is S and whose vertex set is made of all points which belong to an edge in S. By abuse of notation, the subgraph induced by S will also be denoted by S.

We now present the notions of *extension* and *(graph) cut* which play an important role for optimal structures in edge-weighted graphs. The notion of extension was introduced in [4] for the case of sets. In [9, 10] this notion was extended to connected graphs. The following definition presents this notion in the case of unspecified graphs.

Definition 1 (Extension, spanning extension and cut)**.** Let G be a graph and let $G_1, G_2, \ldots, G_n$ be the connected components of G. Let M and X be two subgraphs of G. For any $i \in [1; n]$, let $M_i = M \cap G_i$ and $X_i = X \cap G_i$. We say that X *is an extension of* M if, for all $i \in [1; n]$, $M_i \subseteq X_i$ and each connected component of X_i contains exactly one connected component of M_i. We say that X *is a spanning extension of* M *(over* G*)* if X is an extension of M and if $V(X) = V$. Let $C \subseteq E$, we say that C *is a (graph) cut relative to* M *(over* G*)* if $\overline{C}$ is an extension of M over G and if C is minimal for this property (i.e. considering $D \subseteq E$, $C = D$ whenever $D \subseteq C$ and $\overline{D}$ is an extension of M over G). It may be seen that, if C is a cut, then $\overline{C}$ is necessarily a spanning extension. Moreover, if X is a spanning extension of M, then there exists a unique cut C relative to X which is called the *cut induced by* X. It may be seen that C is also a cut relative to M. □

Examples of these definitions are shown in Figure 1.

2. Optimal structures

In this section we define the following structures: maximum spanning forests, watersheds, minimum cuts and shortest-path spanning forests.

2.1 Maximum spanning forests and watersheds

In this part, we first recall the definition of Maximum Spanning Forests (MaxSF) relative to a subgraph of G. It is shown in [10] that this notion is equivalent to the one of maximum spanning tree, which has been studied for many years in combinatorial optimization (see [8]). From this, we define the MaxSF cut and then remind the notion of watershed to highlight the link that exists between them.

Let F and M be two subgraphs of G. We say that F is a *forest relative to* M if:

- F is an extension of M, and
- for any extension $X \subseteq F$ of M, $V(X) = V(F) \Rightarrow X = F$ (i.e. we cannot eliminate an edge of F and keep the extension property).

Let F and M be two subgraphs of G. We say that F is a *spanning forest relative to* M *(over* G*)* if:

- F is a *forest relative to* M, and
- $V(F) = V$.

Equivalently, we say that F is a *spanning forest relative to* M *(over* G*)* if there exists a spanning extension X relative to M over G such that F is obtained by eliminating edges of X as long as it is possible to do it while preserving the spanning extension property.

Examples of these definitions are shown in Figures 1(e) and 1(f).

It can be seen that if G is connected and $M = (V_M, \emptyset)$ where $V_M \subseteq V$ (i.e. M is a subgraph without edge), then the notion of forest relative to M corresponds exactly to the usual notion of forest. Furthermore, if $|V(M)| = 1$ then we retrieve the usual notion of tree.

In the following, P will be a map from E to $\mathbb{R}^+$.

The pair (G, P) is an *edge-weighted graph*. If e is an edge of G, $P(e)$ is called the *altitude* or the *weight of* e. The *weight of a subgraph* X *of* G, denoted by $P(X)$, is the sum of its edge weights ($P(X) = \sum_{x \in E(X)} P(x)$).

Definition 2 (Maximum spanning forest)**.** Let F and M be two subgraphs of G. We say that F is a *Maximum Spanning Forest (MaxSF)* relative to M (for P) if F is a spanning forest relative to M and if the weight of F is maximum, i.e. greater than or equal to the weight of any other spanning forest relative to M. Notice that if the weight of F is minimum instead of maximum, then we have a *Minimum Spanning Forest (MinSF)*. $\square$

Examples of this definition are shown in Figure 4.

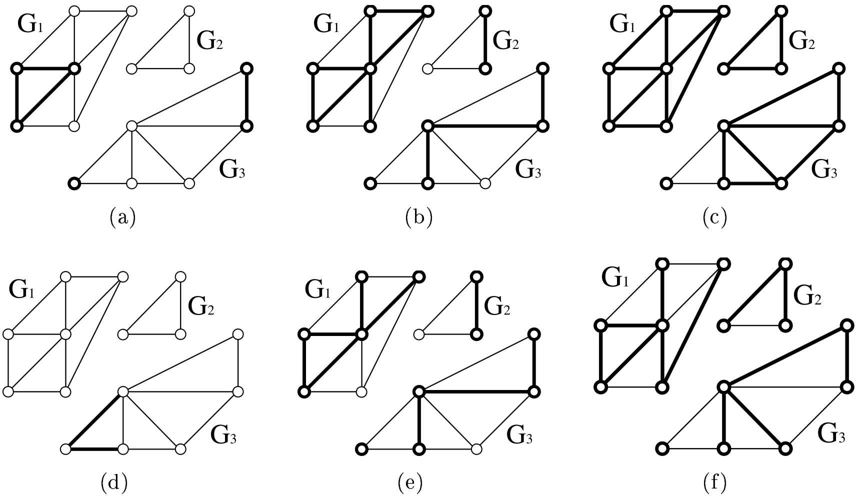


Figure 1. Graph G, composed of three connected components (G_1, G_2 and G_3), with in bold: (a) a subgraph M; (b) an extension relative to M; (c) a spanning extension relative to M; (d) a cut relative to M; (e) a forest relative to M; (f) a spanning forest relative to M.

Remark 1. Let M and F be two subgraphs of G, $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing function and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly decreasing function.
From classical results on extremal spanning forests, we know that the three following statements are equivalent:

- F is a MaxSF relative to M for P;
- F is a MaxSF relative to M for $(f \circ P)$;
- F is a MinSF relative to M for $(g \circ P)$. □

Let M be a subgraph of G and let F be a MaxSF relative to M. Since F is a spanning forest, hence a spanning extension, there exists a unique (graph) cut relative to M induced by F. We say that this cut is a *MaxSF cut relative to M*.

We now remind the definition of watersheds for a map (see [15]) and its equivalence with MinSF cuts relative to the minima of this map (see [9,10]).

The intuitive idea underlying the notion of watershed comes from the field of topography: a drop of water falling down on a topographic surface follows a descending path and reaches a regional minimum area. The watershed may be thought of as the separating lines of the domain of attraction of drops of water.

The regions of a watershed, also called catchment basins, are associated with the regional minima of the map. In other words, each catchment basin contains a unique regional minimum, and conversely, each regional minimum is included in a unique catchment basin: the regions of the watershed are the connected components of an extension relative to the minima. They are separated by a set of edges from which a drop of water can flow down towards different minima, in the sense defined below.

A subgraph X of G is a *(regional) minimum of* P if:

- X is connected, and
- all the edges of X have the same altitude, that we will refer to as the *altitude of* X, and
- the altitude of any edge adjacent to X is strictly greater than the altitude of X.

We denote by $Min(P)$ the graph whose vertex set and edge set are, respectively, the union of the vertex sets and edge sets of all minima of P.

Let $\pi = \langle x_0, \ldots, x_\ell \rangle$ be a path in G. The path π is descending (for P) if $\forall i \in [1, \ell - 1], P(\{x_{i-1}, x_i\}) \geq P(\{x_i, x_{i+1}\})$.

Definition 3 (Watershed, Def. 2.3 in [9])**.** Let C be a subset of E. We say that C is a *watershed cut (for* P*)*, or simply a *watershed (for* P*)*, if $\overline{C}$ is an extension of $Min(P)$ and if for any $e = \{x_0, y_0\} \in C$, there exist $\pi_1 = \langle x_0, \ldots, x_m \rangle$ and $\pi_2 = \langle y_0, \ldots, y_n \rangle$ two descending paths in $\overline{C}$ such that:

- x_m and y_n are nodes of two distinct minima of P, and
- $P(e) \geq P(\{x_0, x_1\})$ (resp. $P(e) \geq P(\{y_0, y_1\})$), whenever $m > 0$ (resp. $n > 0$). □

Notice that a watershed is indeed a graph cut relative to $Min(P)$.

Theorem 1 (Th. 3.1 in [9])**.** *Let* C *be a subset of* E*. The set* C *is a MinSF cut relative to* $Min(P)$ *(for* P*) if and only if* C *is a watershed (for* P*).*

Any minimum spanning tree algorithm can be employed to compute a MinSF relative to a subgraph of G (see a survey in [8]). The best of them does this in quasi-linear time (see [7]), but algorithms specific to watersheds run in linear time (see [9]).

2.2 Minimum cuts (min-cuts)

In this section, we remind the notion of minimum cut.

Let M be a subgraph of G and let $C \subseteq E$. We say that C *is a minimum cut (min-cut) relative to* M *(for* P*)* if for any cut $C' \subseteq E$ relative to M, $P(C) \leq P(C')$. It can be seen that a cut C relative to M is of minimum weight if and only if $\overline{C}$ is a (spanning) extension of maximum weight relative to M. Examples of this definition are shown in Figure 4.

A fundamental result in combinatorial optimization states that, given two isolated nodes of an edge-weighted graph (called source and sink), finding a min-cut that separates these two nodes is equivalent to finding a maximum flow between them (see [12], chapter 6.2). This problem is equivalent to finding a min-cut relative to a subgraph having exactly two connected components (consider adding two extra nodes to G, the source and the sink, and highly weighted edges from each one of them to all the nodes of each of the components of M). In this case, we have polynomial-time algorithms to compute a min-cut. On the other hand, finding a min-cut relative to a subgraph with more than two connected components is NP-hard [11], but there exists approximation algorithms [6].

2.3 Shortest-path spanning forests cuts (SPSF cuts)

We now present the notion of shortest-path forest which also constitutes an optimization paradigm used for image segmentation. In particular, the image-foresting-transform [13] and the relative fuzzy-connected image segmentation [2, 17] fall in the scope of shortest-path forests. Intuitively, these methods partition the graph into connected components associated to seed points. The component of each seed consists of the points that are "more closely connected" to this seed than to any other. In many cases, in order to define the relation "is more closely connected to", we consider the length of a path π as the maximum value of an edge along π. Then, point p is more closely connected to seed s than to seed s' if the length of a shortest path from p to s is less than the length of a shortest path from p to s'. Given a set of seed points (or a seed graph), the resulting segmentation is then obtained as a shortest-path forest.

In this section, we assume that G is connected and that M is non-empty.

Let $\pi = \langle x_0, \ldots, x_\ell \rangle$ be a path in the graph G. If we have $l > 0$, we define $P(\pi) = \max\{P(\{x_{i-1}, x_i\}) \mid i \in [1;\ell]\}$. If we have $\pi = \langle x_0 \rangle$, we define $P(\pi) = \min\{P(u) \mid x_0 \in u, u \in E\}$; $P(\pi)$ is the *length of* π. Let X and Y be two subgraphs of G, we denote by $\Pi(X,Y)$ the set of all paths from X to Y in G. The *connection value between X and Y (in G and for P)*, denoted by $P(X,Y)$, is the length of a shortest path from X to Y, i.e. $P(X,Y) = \min\{P(\pi) \mid \pi \in \Pi(X,Y)\}$.

If x is a vertex of G, to simplify the notation, the graph $(\{x\}, \emptyset)$ will be also denoted by x.

Definition 4 (SPSF cut)**.** Let M and F be two subgraphs of G. We say that F is a *shortest-path forest relative to M* if F is a forest relative to M and if, for any $x \in V(F)$, there exists, from x to M, a path π in F such that $P(\pi) = P(x,M)$. If F is a shortest-path forest relative to M and $V(F) = V$, we say that F is a *shortest-path spanning forest (SPSF) relative to M*. If F is a SPSF relative to M, the (unique) cut for F is called a *SPSF cut for M*. □

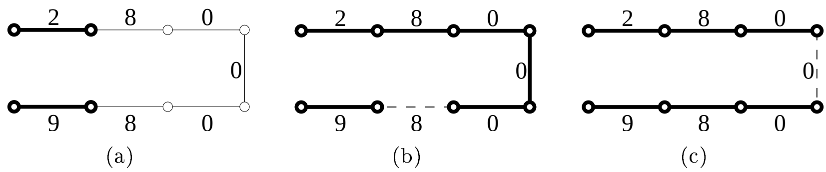


Figure 2. Graph G and map P with in bold: (a) a subgraph M; (b) a MinSF relative to M; (c) a shortest-path spanning forest relative to M which is not a MinSF relative to M.

Let G be the graph in Figure 2 and let P be the corresponding map. Let M, F and F' be the bold graphs depicted in, respectively, Figures 2(a), 2(b) and 2(c). The two graphs F and F' are SPSFs relative to M. The induced SPSF cuts for M are represented by dashed edges.

3. Some links between optimal structures

In this section, we reveal some relations existing between the different optimal structures exposed above.

3.1 Min-cut and MaxSF cut

In this section, we show that min-cuts and MaxSF cuts are linked through a modification of the map P preserving the order and emphazing the weight difference between the edges. We denote by $P^{[n]}$, and say P power n, the map from E to $\mathbb{R}^+$ defined by, for any $e \in E$, $P^{[n]}(e) = [P(e)]^n$.

Theorem 2. *If M is a subgraph of G, then there exists a real number m such that, for any $n \geq m$, any min-cut relative to M for $P^{[n]}$ is a MaxSF cut relative to M for $P^{[n]}$.*

Theorem 2 is illustrated in Figure 3 and Figure 4.

It has to be noticed that the converse of Theorem 2 is, in general, not true. See Figure 6 where the MaxSF cut relative to M for P in 6(b) is not a min-cut relative to M for P, but any min-cut is a MaxSF cut. However, an intuitive interpretation of this result is to consider the MaxSF cut as a greedy heuristic to obtain a min-cut. The efficiency of this heuristic becomes higher when differences between the weights increase.

From Remark 1, we know that the MaxSF cut relative to M for $P^{[n]}$ is also a MaxSF cut relative to M for P and conversely since the change of map preserves the order.

Since we know, from Remark 1 and Theorem 1, that the watersheds are particular cases of MaxSF cuts, we deduce from Theorem 2 that the watersheds are also particular cases of the min-cuts.

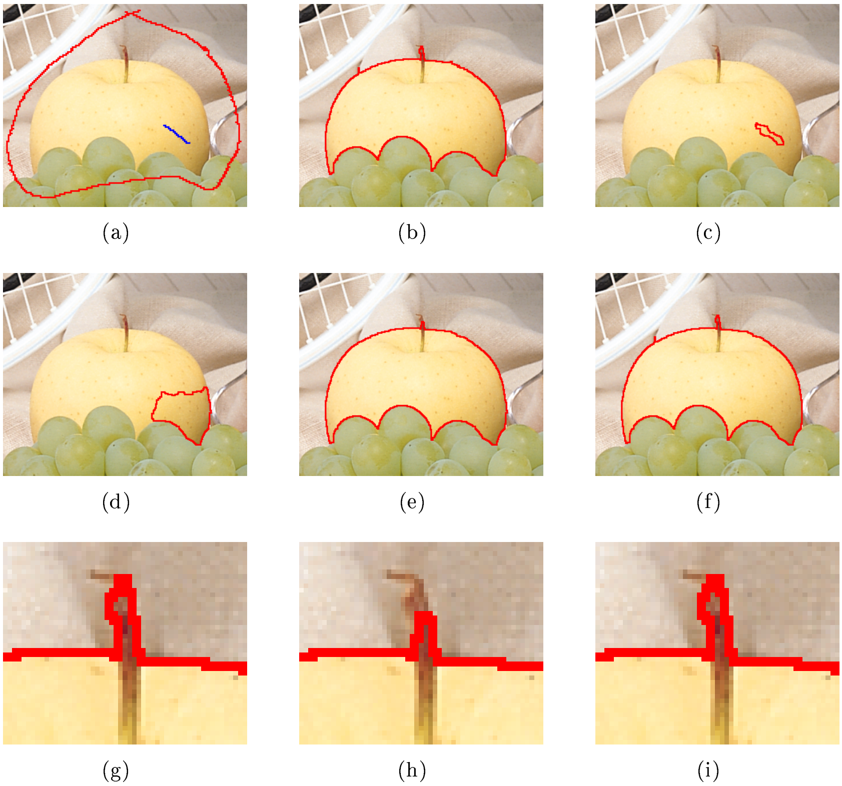


Figure 3. Color image segmentation using: (a) markers superimposed to the original image; (b) watershed on P; (c) min-cut on P; (d) min-cut on $P^{[1.4]}$; (e) min-cut on $P^{[2]}$; (f) min-cut on $P^{[3]}$; (g) zoom of watershed on P; (h) zoom of min-cut on $P^{[2]}$; (i) zoom of min-cut on $P^{[3]}$.

Figure 3 illustrates the link between these two well known segmentation paradigms through the evolution of the min-cut with different values of n. Notice that the power of the map P could then be considered as a smoothing term for the min-cut method. Indeed, when this power decreases, shortest cuts are found whereas, when it increases, longer cuts are found. These longer cuts can surround more details as well as noise. Therefore, releasing this smoothing term is not always suitable. See for example Figure 5 where the min-cut result is better than the watershed.

3.2 MinSF cuts and SPSF cuts

We now investigate the links between SPSF cuts and MinSF cuts. We show that any MinSF cut relative to a subgraph of G is a SPSF cut relative to this

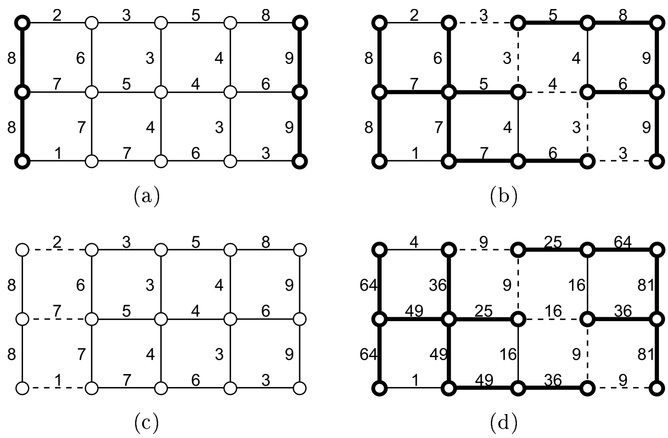


Figure 4. Graph G and map P with: (a) in bold, a subgraph M; (b) in bold, the MaxSF relative to M for P and, in dashed edges, its induced cut which, according to Remark 1 and Theorem 1, is a watershed (up to a strictly decreasing function on P); (c) in dashed edges, the min-cut relative to M for P; (d) in bold, the MaxSF relative to M for $P^{[2]}$ and, in dashed edges, its induced cut which is also the min-cut relative to M for $P^{[2]}$.

subgraph. Therefore, according to Theorem 2, there exist some particular functions for which any min-cut is a SPSF cut (up to a strictly decreasing function over P). Furthermore, we prove that MinSF cuts and SPSF cuts are equivalent whenever we consider the subgraph of G which corresponds precisely to the minima of P. Hence, according to Theorem 1, this last result establishes the equivalence between the watersheds for P and the SPSF cuts relative to the minima of P.

In this section, we assume that G is connected and that M is non-empty.

Theorem 3 (Prop. 30 in [10]). *Let M and F be two subgraphs of G. If F is a MinSF relative to M, then F is a shortest-path forest relative to M. Furthermore, any MinSF cut relative to M is a SPSF cut relative to M.* [1]

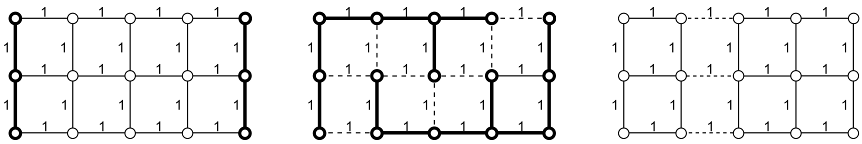
The converse of the previous theorem is, in general, not true. For instance, the graph Z (Figure 2(c)), is a SPSF relative to the graph X (Figure 2(a)) whereas it is not a MinSF relative to this graph.

In fact, as stated by the following theorem, if the graph M constitutes precisely the minima of P, the equivalence between both concepts can be established.

[1] This result was obtained independently in [3].

(a) (b) (c)

Figure 5. Color image segmentation: (a) markers superimposed to the original image; (b) watershed on P; (c) min-cut on P.



(a) (b) (c)

Figure 6. Graph G and map P with: (a) in bold, a subgraph M; (b) in bold, a MaxSF relative to M for P and, in dashed edges, its induced cut, which is not a min-cut; (c) in dashed edges, a min-cut relative to M for P.

Theorem 4 (Prop. 31 in [10])**.** *Let F be a subgraph of G. The graph F is a SPSF relative to $Min(P)$ if and only if F is a MinSF relative to $Min(P)$. Furthermore, a cut S relative to $Min(P)$ is a SPSF cut relative to $Min(P)$ if and only if S is a MinSF cut relative to $Min(P)$.*

Conclusion

We compared three different optimal structures, namely extremal spanning forests, min-cuts and shortest-path forests, which have been used as the basis for popular image segmentation methods. The watershed approach, which is strongly linked to minimum spanning forests and to shortest-path forests, is also considered in this study. Although different in general, we exhibited some particular cases where a strong relation exists between these structures.

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