Vision 3D artificielle - Final exam
(duration: 2h30)

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The exercises are independent. You can choose to answer in French or English, at your convenience.

1 Homographies and Conics

A conic is defined as a set of points $a \in \mathbb{P}^2$ such that $a^\top C a = 0$ for a given $C \in \mathbb{R}^{3\times 3}$, an invertible, symmetric matrix. Its type (resp. ellipse, parabola or hyperbola) depends on the sign of the determinant of its 2×2 upper-left submatrix $|C_{1:2,1:2}|$ (resp. $\geq$, $=$ or ≤ 0). The goal of this exercise is to see how conics can help determine a homography.

1.1 Preliminaries

1. From the formulas of the course, deduce the following identities ($a, b, c, d \in \mathbb{P}^2$):

$$(a \times b)^\top (c \times d) = a^\top c b^\top d - a^\top d b^\top c, \qquad (a \times b) \times (c \times d) = \begin{vmatrix} a & b & d \end{vmatrix} c - \begin{vmatrix} a & b & c \end{vmatrix} d. \tag{1}$$

1.2 Conics

2. Justify why a given conic is defined by C but also by any $C' = \lambda C$ with $\lambda \in \mathbb{R} \setminus \{0\}$.

3. Write a matrix C for each of the following conics ($x_0, y_0 \in \mathbb{R}$, $a, b \in \mathbb{R} \setminus \{0\}$):

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 \tag{2}$$

$$y - y_0 = a(x - x_0)^2 \tag{3}$$

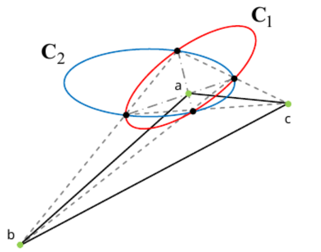
$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1 \tag{4}$$

4. Show that from 5 points in general position, there is a unique ellipse containing them (you are not required to check the equations are independent) and propose a numeric procedure to recover the matrix C.

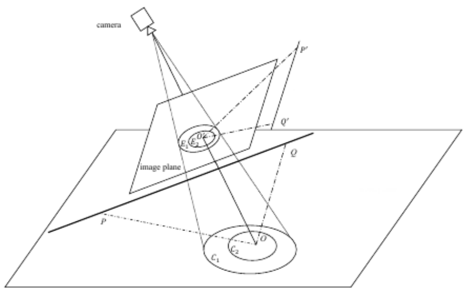
5. Show that if $a \in \mathbb{P}^2$, then the line ℓ represented by the vector Ca is the tangent to the conic C at a. Hint: suppose that another point $b \neq a$ of ℓ is on the conic, write three equations with the data and arrive at a contradiction $a = b$.

1.3 Self-Polar Triangles

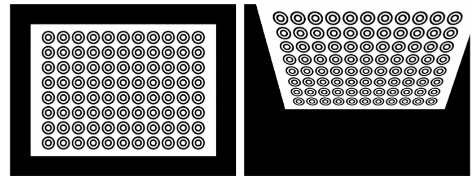
6. We call polar (relative to conic C) associated to a point (or pole) a the line represented by vector Ca. A triangle of 3 disinct points abc is called *self-polar* if the polar of a (resp. b, c) is the line through b and c, (resp. a and c, a and b). Show that a self-polar triangle is degenerate (points aligned) if and only if they are on a tangent to C.



Question 12: common self-polar triangle of conics with 4 intersection points.



Question 18: image of a ring through a camera.



Question 19: views of a calibration rig made of rings.

Figure 1: Illustrations for the first exercise

7. Show that given a not on the conic, there is a self-polar triangle whose one vertex is a.
8. Show that for a non-degenerate self-polar triangle, we can write ($\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$)

$$Ca = \alpha\, b \times c \qquad\qquad Cb = \beta\, c \times a \qquad\qquad Cc = \gamma\, a \times b. \tag{5}$$

9. A point is *inside* an ellipse if and only if $|C|\, a^\top C a > 0$. Show that if a is inside the ellipse, then all points of the polar line Ca are outside: take b on Ca and $c = a + \lambda b$, then compute $c^\top C c$.
10. Show that in a non-degenerate self polar triangle, one vertex is inside and the other two are oustide the ellipse. For that, consider the determinant $|Ca \quad Cb \quad Cc|$.
11. Let two distinct conics C_1 and C_2 having a common self-polar triangle, show its vertices are eigenvectors of the matrix $C_1^{-1} C_2$.
12. Prove that if two conics have four distinct common points, the three points obtained by taking the intersections of the line going through the intersection of two common points and of the line going through the other two is a common self-polar triangle (see Figure 1).

1.4 Conics through Homography

13. Show that a homography transforms a conic to a conic.
14. Show that a homography transforms the three vertices of a self-polar triangle of a conic to a self-polar triangle of its image by the homography.
15. Show that the center of symmetry of an ellipse or hyperbola has Euclidean coordinates $(C_{1:2,1:2})^{-1} C_{1:2,3}$ (Matlab notations).
16. Show that an affine transform maps the center of symmetry of an ellipse to the center of symmetry of the mapped ellipse, which may not be true for a general homography (give a counter-example).
17. Given two concentric circles of matrices C_1 and C_2, find a common self-polar triangle.
18. Deduce a mean to determine the image of the common center by a homography (see Figure 1).
19. From a planar pattern of circular rings, propose an algorithm (in pseudo-code) to calibrate a camera (see Figure 1).

2 Graph cuts for disparity estimation

To handle occlusion in diparity estimation, we call assignment a pair $a = (p, q)$ of pixels in the left and right (stereo-rectified) images u_L and u_R, the disparity being $d(a) = x_q - x_p$. We attach a binary label 0 (inactive) or 1 (active) to each assignment. Such a labelization is said admissible if for any p and q at most one assignment $(p, .)$ is active, so as at most one assignment $(., q)$. If none, the pixel is considered occluded, otherwise it gets the disparity $d(a)$ of its active assignment. We consider a graph where nodes are all possible assignments, that is, all assignments $a = (p, q)$ such that $d(a) \in [d_{\min}, d_{\max}]$, the interval of possible disparities for all pixels.

20. To recover an admissible labelization, we put infinite (or very large) cost to edges between assignments sharing either p or q. Explain why recovering the labelization is not possible exactly by graph cuts, despite it being a binary labelization problem.
21. For each active assignment $a = (p, q)$ we pay a cost $\omega(a) = \|u_L(p) - u_R(q)\|$ (an arbitrary colorspace norm); for two active assignments $a = (p, q)$ and $a' = (p', q')$ with $d(a) \neq d(a')$ whereas p and p' (or q and q') are 4-neighbors, we pay a fixed cost $\lambda > 0$ if $\max(\|u_L(p) - u_L(p')\|, \|u_R(q) - u_R(q')\|) \geq \mu$ ($\mu > 0$ a fixed parameter)) and a cost $\lambda' > \lambda$ otherwise. Explain the reason for these choices.
22. Explain why we need to put a cost $K > 0$ for each inactive assignment.
23. Write the formula of the energy for such a labelization.
24. Given α a possible disparity, remind what is an α-expansion.
25. Construct a weighed graph with nodes some relevant assignments such that the optimal α-expansion from an initial admissible labelization may be found exactly by graph cut.
26. Propose as pseudo-code the algorithm for approximating the minimal energy labelization.