COMPARISON OF BEM AND FEM METHODS FOR THE E/MEG PROBLEM

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ABSTRACT

The direct E/MEG problem consists of simulating the electromagnetic field produced by neuronal electric sources on the cortex. We compare two different methods for the resolution of this problem, mainly from the point of view of computational complexity and accuracy. First, the finite element method (FEM), based on the discretization of the PDE in the entire head volume. Second, the boundary element method (BEM), discretizing the equivalent integral equations on the surfaces separating volumes with different electrical parameters. We also study the behaviour of the BEM and FEM methods for the sources approaching the discontinuity in conductivity, as it makes the accuracy of the solution decrease.

We conclude that at the current state of investigation, for equivalent triangulation/tetraheaderisation, the FEM is significantly faster than BEM and provides similar or better accuracy. It should therefore be used whenever the volumic tetraheadric mesh is available.

1. INTRODUCTION

Electro- and magneto-encephalography (E/MEG) [1] is a non-invasive diagnostic and observational technique potentially capable of studying brain activity at much shorter time scales than other alternatives (fMRI, SPECT, or PET). Neuronal activity inside the cortex is related to movements of charged ions. These so called *primary currents* (sources) in turn provoke weak electric and magnetic field outside the subject's head, which can be measured by electrodes for electric field and superconductive quantum magnetic sensors (SQUIDs) for the magnetic field. The relation between the primary currents and the measurements can be derived from Maxwell and material equations [**?**]. This is essential for the *inverse problem* (determining the sources from the measurement) to improve the localization accuracy of the E/MEG, which is for the moment significantly worse than that of the above mentioned alternative methods. However, a plausible model of the geometrical and electrical properties of the head needs to be sufficiently refined. This leads to large to very large data-sets. We estimated in [**?**] that for accurate modelisation of the cortex approximately $10^5 \sim 10^6$ surface elements would be needed, which corresponds to probably $10^7 \sim 10^8$ volume elements. Consequently, the choice of the mathematical formulation of the physical problem and its associate numerical method is very important and can have a major impact on the feasibility, accuracy, and speed of the approach.

We want to allow an arbitrary distribution of primary current sources, thus putting aside methods such us those trying to identify for example a small (fixed) number of current dipoles. This leaves us several large classes of methods, the two most important ones being differential, volume based finite element methods (FEM) and integral, surfase based boundary element methods (BEM).[1] However, the literature gives no indication about which of the two should be used for the problem of electromagnetic modeling of the brain. As far as we know, general comparison is not available. Same authors claim that BEM is more advantageous than FEM only for problems of infinite extents or unknown or moving boundaries [**?**], while others observe that it delivers significantly better precision for equivalent grids [**?**]. The aim of this article is therefore to provide a meaningful comparison of one specific implementation of the FEM and the BEM for the application at hand.

In this article, we concentrate on the *direct problem*: to determine the external field given the primary sources. The results are much easier to interpret and the relation to the behavior for the inverse problem should be straightforward. We also focus on synthetic examples where the analytical solution (the ground truth) is available and can be used to evaluate the accuracy of a method.

Connect the MEG to EEG. Say that we concentrate on the EEG. EEG and MEG come from Maxwell equations. The equations are obtained using a quasi-static approximation. Biot-Savart ?

[1]The third group consisting of much less used finite difference methods will not be dealt with here.

2. FINITE ELEMENT METHOD

In this section, the head domain and its boundary are denoted by the names Ω and S, respectively. In practice, part of S will contain the scalp. (figure ?) The direct EEG problem corresponds to compute the electric potential $V(\mathbf{r})$ over Ω, given a known source distribution $\mathbf{J}^p$. This potential satisfies the following state equation and boundary condition:

$$\begin{cases} \nabla \cdot (\sigma \nabla V) &= \nabla \cdot \mathbf{J}^p \quad \text{in} \quad \Omega \\ \dfrac{\partial V}{\partial \mathbf{n}} = \nabla V \cdot \mathbf{n} &= g \qquad \text{on} \quad S = \partial\Omega. \end{cases} \tag{1}$$

This PDE (Partial Differential Equation) is reformulated so as its solution is obtained by minimizing the energy function:

$$E(V) = \frac{1}{2} \int_\Omega \sigma(\mathbf{r}) |\nabla V(\mathbf{r})|^2 d\mathbf{r} - \int_\Omega \nabla \cdot \mathbf{J}^p(\mathbf{r}) V(\mathbf{r}) d\mathbf{r} \ , \tag{2}$$

over all the functions V satisfying the boundary condition. This continuous integral problem is solved using the Galerkin P^1 method which approximates the solution V by a piece-wise affine function over a tetrahedral mesh. Thus, denoting by $\mathbf{r}_i, i = 1..N$ the mesh vertices, V can be written as $V(\mathbf{r}) = \sum_{i=1}^N V_i \phi_i(\mathbf{r})$, where $\phi_i(\mathbf{r}_j) = \delta_{ij}$ is a set Φ of base functions for the piece-wise linear function over the mesh.. The potential is thus given by its values at each vertex $\mathbf{r}_i$ of the mesh, which we collect into a vector $\mathcal{V}$. Then, the solution for Eq. 2 can be approximated by solving the linear system: $\mathcal{A}\mathcal{V} = \mathcal{B}$, where the elements A_{ij} and B_i of the $N \times N$ matrix $\mathcal{A}$ and the vector $\mathcal{B}$ of size N are given by:

$$\begin{aligned} A_{ij} &= \int_\Omega \sigma(\mathbf{r}) \nabla \phi_i(\mathbf{r}) \cdot \nabla \phi_j(\mathbf{r}) d\mathbf{r} \\ B_i &= \int_\Omega \nabla \cdot \mathbf{J}^p(\mathbf{r}) \phi_i(\mathbf{r}) d\mathbf{r} \end{aligned}$$

In practice, the source distribution $\mathbf{J}^p$ and the conductivity σ are also discretized over the base Φ so that these integrals can be pre-computed given the mesh, and the conductivities and primary source currents at the mesh vertices. To make the comparison with the BEM methods, piece-wise constant conductivities is the model adopted in this paper, but the previous model is not limited to this simple case. This is one theoretical advantage of the method when compared to the BEM method. An important property of the matrix is that it is symmetric and very sparse (as most of the functions ϕ_i do not share the same support), so that the system can be solved efficiently using a conjugate gradient descent with optimal step.

2.1. FEM computational complexity

[Il vaut mieux que ça soit Théo qu'il écrive...]

How many operations as a function of the element size, how does an elementary operation look like, what can be precalculated, convergence speed of the iterative solver.

Storage requirements.

2.2. FEM acceleration

Sparsity, multiscale / multigrid.

3. BOUNDARY ELEMENT METHOD

The field equations [reference] have several alternative integral forms derived from Green equivalencies ($\int_\Omega \nabla \cdot \mathbf{f} \mathrm{d}V = \int_{\delta\Omega} \mathbf{f} \cdot \mathbf{n} \mathrm{d}S$) An advantage of the integral forms is that they involve the unknown quantity (for example the electric potential) only on a surface, resp. several surfaces. Once the surface values is known, the field in the rest of the volume can be then directly calculated from the appropriate integral expressions. This reduces the the dimensionality of the problem (recall that the FEM has unknowns distributed in the whole volume) and consequently also the number of unknowns. Moreover, only a surface meshing is now necessary.

We have chosen the often used [2, 3] integral formulation

$$\begin{aligned} \sigma_0 V_0(\mathbf{r}) = \frac{\sigma_m^+ + \sigma_m^-}{2} V(\mathbf{r}) + \\ \sum_{k=1}^N \frac{\sigma_k^- - \sigma_k^+}{4\pi} \int_{\mathbf{r}' \in S_k} V(\mathbf{r}') \nabla' \left(\frac{1}{R} \right) \cdot \mathbf{n}_k(\mathbf{r}') \, \mathrm{d}s_k(\mathbf{r}') \end{aligned} \tag{3}$$

involving the potential $V(\mathbf{r})$ on a smooth surfaces S_m separating regions with conductivities σ_m^+, σ_m^-, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $R = \|\mathbf{R}\|$ and $\nabla' \left(\frac{1}{R}\right) = \frac{\mathbf{R}}{R^3}$, where N is the number of surfaces, $\mathbf{n}_k$ is the unit vector normal to S_k oriented from the region with σ_k^- to the one with σ_k^+. V_0 is a potential due to a known primary current distribution $\mathbf{J}^p(\mathbf{r})$ in the infinite homogeneous space with conductivity σ_0 [2]

$$\sigma_0 V_0(\mathbf{r}) = \frac{1}{4\pi} \int_{\mathbf{r}' \in \mathbb{R}^3} \mathbf{J}^p(\mathbf{r}') \cdot \nabla' \left(\frac{1}{R} \right) \mathrm{d}\mathbf{r}' \ . \tag{4}$$

The potential V is discretized using either piecewise constant (P0) or piecewise linear (P1) basis functions φ_i each associated with a surface element (for example a triangle). The continuous integral equation (3) is converted into a corresponding linear equation system by taking a scalar product with test functions ψ_j which are normally chosen to be either the Dirac's δ (the collocation method), or $\psi_j = \varphi_i$ (the Galerkin method). In both cases we obtain a linear

equation system $\mathsf{A}\mathbf{v} = \mathbf{b}$ relating $\mathbf{b}$ representing V_0 and the unknown coefficients $\mathbf{v}$ of the discretization of V. The elements of the matrix A are given by integrals of the type

$$\Gamma_{i,j}^{\mathbf{n}_i\varphi_i,\psi_j} = \iint_{\substack{\mathbf{r}\in\operatorname{supp}\psi_j \\ \mathbf{r}'\in\operatorname{supp}\varphi_i}} \nabla'\left(\tfrac{1}{R}\right)\cdot\mathbf{n}_i\,\varphi_i(\mathbf{r}')\psi_j(\mathbf{r})\mathrm{d}s^2(\mathbf{r}',\mathbf{r}) \qquad (5)$$

We note that A is non-symmetric and full.

3.1. BEM computational complexity

We discretize the surfaces S_m with elements of a characteristic size h. Then the number of elements N grows as $O(h^{-2})$. The number of unknowns (v_i) is typically N (or kN for some small k for higher order approximations), the number of equations at least as much. Consequently, the size of A is $O(h^{-2}) \times O(h^{-2})$ and the cost of calculating all its elements $O(h^{-4}) = O(N^2)$, equal as the memory consumption. A direct method (such as LU decomposition) solves the associated linear system in $O(h^{-6}) = O(N^3)$ time. Moreover, the evaluation of the integrals (5) is involved even for basis functions φ and ψ of very small orders, featuring computationally expensive functions such as square roots and inverse tangents [4]. For higher orders (even as low as $\psi = P0$ and $\phi = P0$), numerical integration (typically Gaussian quadrature) needs to be used at least for one of the integrals. Because of the singularity of the kernel at $R = 0$, the number of points of integration at least for close elements has to be high, typically around $30 \sim 100$ points per element for an acceptable level of precision ($10^{-3} \sim 10^{-5}$, depending on the geometry. We see that the resulting time and memory complexity of the direct implementation is clearly prohibitive for the size of the problems we want to consider.

3.2. BEM acceleration

The direct solution of the linear system with its associated fixed-cost $O(N^3)$ can be replaced by an iterative method (such as GMRES [**?**]) that accesses the matrix A only by matrix-vector products $\mathsf{A}\mathbf{x}$. This enables us to avoid the excessive storage by calculating the matrix elements on-the-fly. In many cases, the iterative method can provide results with only a small number of iterations m, which reduces the computational complexity to $O(mN^2) = O(mh{-}4)$. The matrix-vector product complexity $O(N^2)$ can be reduced up to $O(N \log N)$ by fast multipole method (FMM) [**?**] like techniques that hierarchically approximate the interaction of a group of elements by a single term. This brings the total asymptotic complexity down to $O(mN \log N) = O(-mh{-}2\log h)$. Nevertheless, the cost of preprocessing increases significantly.

3.3. Sources close to discontinuities

It is known that the accuracy of all existing implementations of the BEM suffers greatly when the current source approaches the discontinuity [**?**, 3, 5, 6]. This is because the potential V varies strongly along the surface when the source is close and although the approximation in the basis φ_i is acceptable in the ordinary sense, it becomes very bad when multiplied by the highly varying kernel in (5). This explains also the somewhat cotraintuitive result that in some situations the seemingly primitive collocation method with P0 elements can outperform higher order methods, precisely because the approximation error is small at the collocation points where it is given most weight. Note the imprecision is in the matrix A and is therefore independent of the source configuration.

4. EXPERIMENTS

We have performed the experiments using a simplified head model consisting of three concentric spheres with diameters 0.87, 0.92 and $1.0\,\mathrm{m}$ delimiting domains of conductivities 1.0, 0.0125, 1.0, and $0\,\Omega^{-1}\mathrm{m}^{-1}$, representing respectively the brain, the bone, the scalp, and the outside air [**?**]. The heads were approximated with progressively finer surface meshes, using 80, 320, 1280, and 5120 triangles per sphere; the surface meshes were extended to corresponding volume meshes.

A source is a current dipole $[1\ 0\ 1]/\sqrt{2}$ placed between $50 \sim 97\,\%$ of the radius of the innermost sphere. The potential for this configuration was calculated analytically [7, **?**] at the vertices and centres of the triangles, an example is shown in Figure 1.

The best performing BEM (P0 elements, collocation, P0 weighting of the V_0 term, direct solution for small sizes, single level FMM and iterative solution using GMRES for bigger number of elements) and FEM (P1, iterative) were used in the experiments.

Figure 2 shows the dependence of the mean (ℓ_2) and maximum (ℓ_∞) relative errors on the position of the dipole for the four head sizes. We see that while for both methods the precision improves with the number of elements and much faster for BEM, it is almost always better for the FEM. The precision decreases as the dipole approaches the 'cortex' surface (Figure 3), however in BEM the effect is much more pronounced.

Finally, Table 1 summarizes the execution time[2] for the BEM using direct linear system solution and the single level FMM, and for the FEM. We see that while for small problems the direct BEM implementation is faster, for bigger problems it pays off to use the FMM.

[2] Pentium III computer at 1000 MHz with 1024 MB of RAM was used.

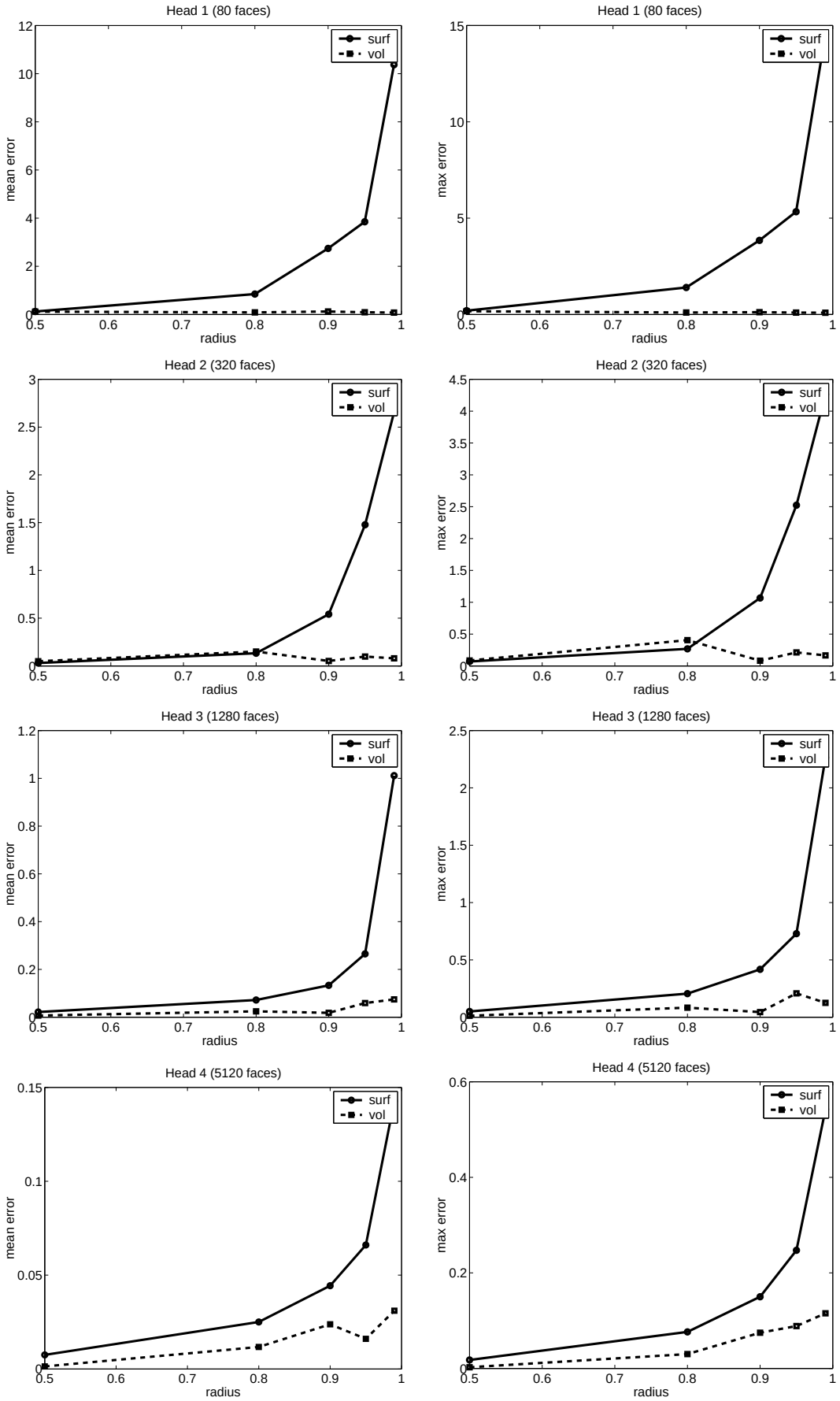

Fig. 2. The evolution of the mean and maximum relative errors as a function of the dipole position for the BEM and FEM for various mesh sizes.

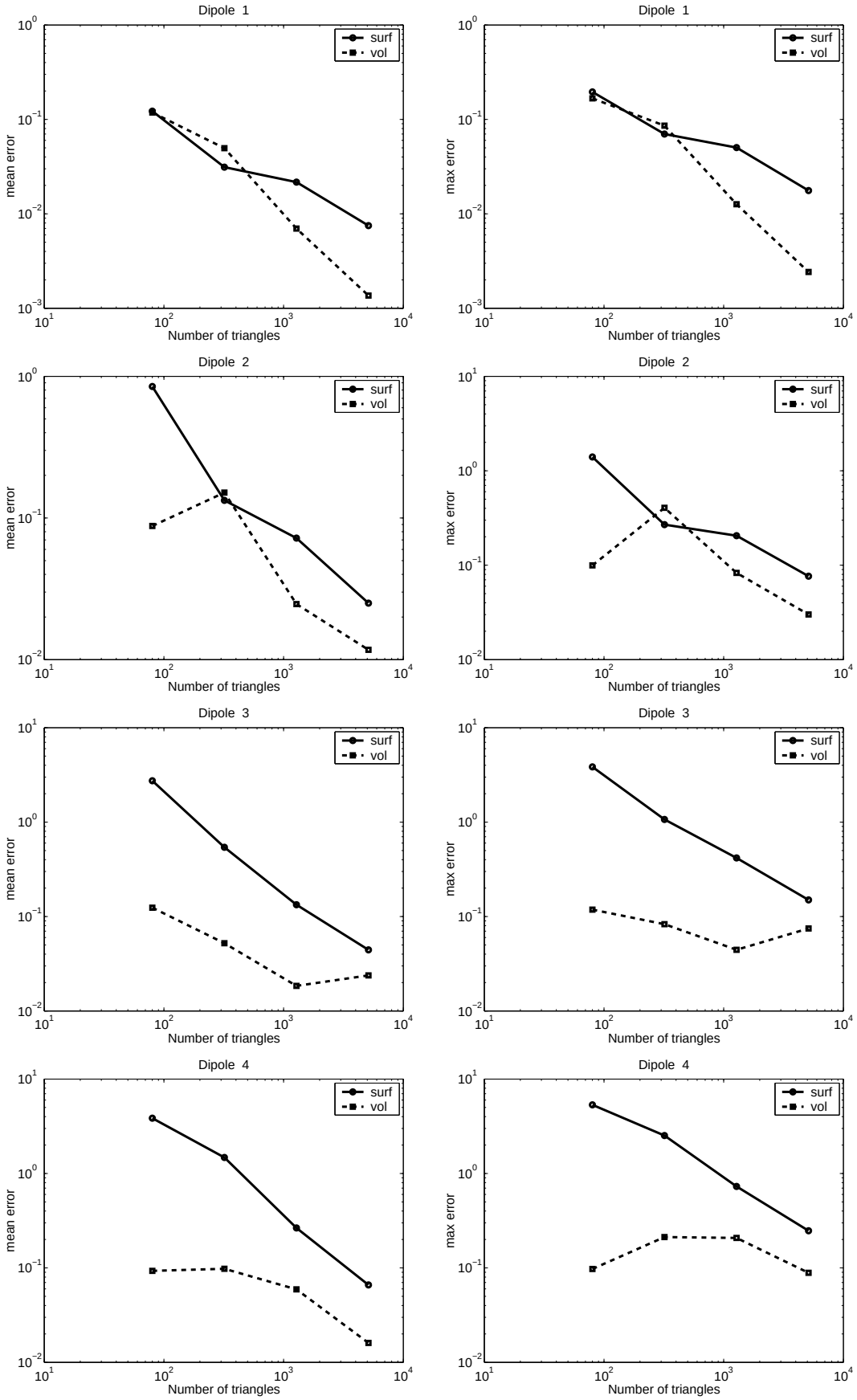

Fig. 3. The evolution of the mean and maximum relative errors as a function of the number of surface elements for the BEM and FEM for dipole number 3 (at 88 % of the radius).

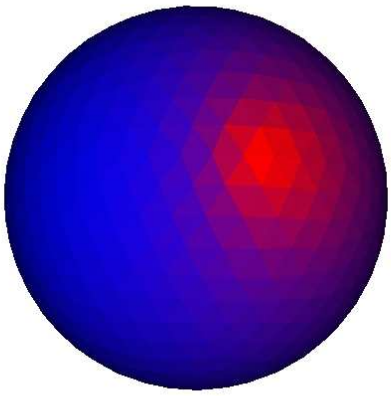

Fig. 1. Analyticaly calculated potential on the surface of the outermost sphere represented by 1280 triangles due to a dipole at $88\,\%$ of the radius.

Table 1. Typical execution times for the FEM and two variants of the BEM — direct implementation and an implementation using FMM — for varying model sizes. N_{surf} is the number of surface elements. The direct method ran out of memory for the largest problem.

N_{surf}	80	320	1280	5120
BEM direct	0.2 s	16.2 s	21 m	—
BEM + sFMM	4.1 s	34.8 s	22 m	1 h 23 m
FEM				

5. CONCLUSIONS

The BEM works with less elements than the FEM but the number of interactions to consider is higher. Using FMM the number of interactions to actually calculate can be reduced significantly but but the cost of calculating one interaction will be always higher than for the FEM. Most importantly, the BEM is plagued by inherent accuracy problems when the sources approach the discontinuity. This could be perhaps remedied by a different integral formulation but none is known for the moment. Moreover, FEM can accommodate for continuously varying and anisotropic conductivities.

The only significant disadvantage of FEM is the necessity of creating the 3D volumic mesh, which is apparently much more difficult than creating the surface mesh. According to our experiments, if the volumic mesh is available, then with the current state of art, it seems preferable to use the FEM for E/MEG applications.

6. REFERENCES

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