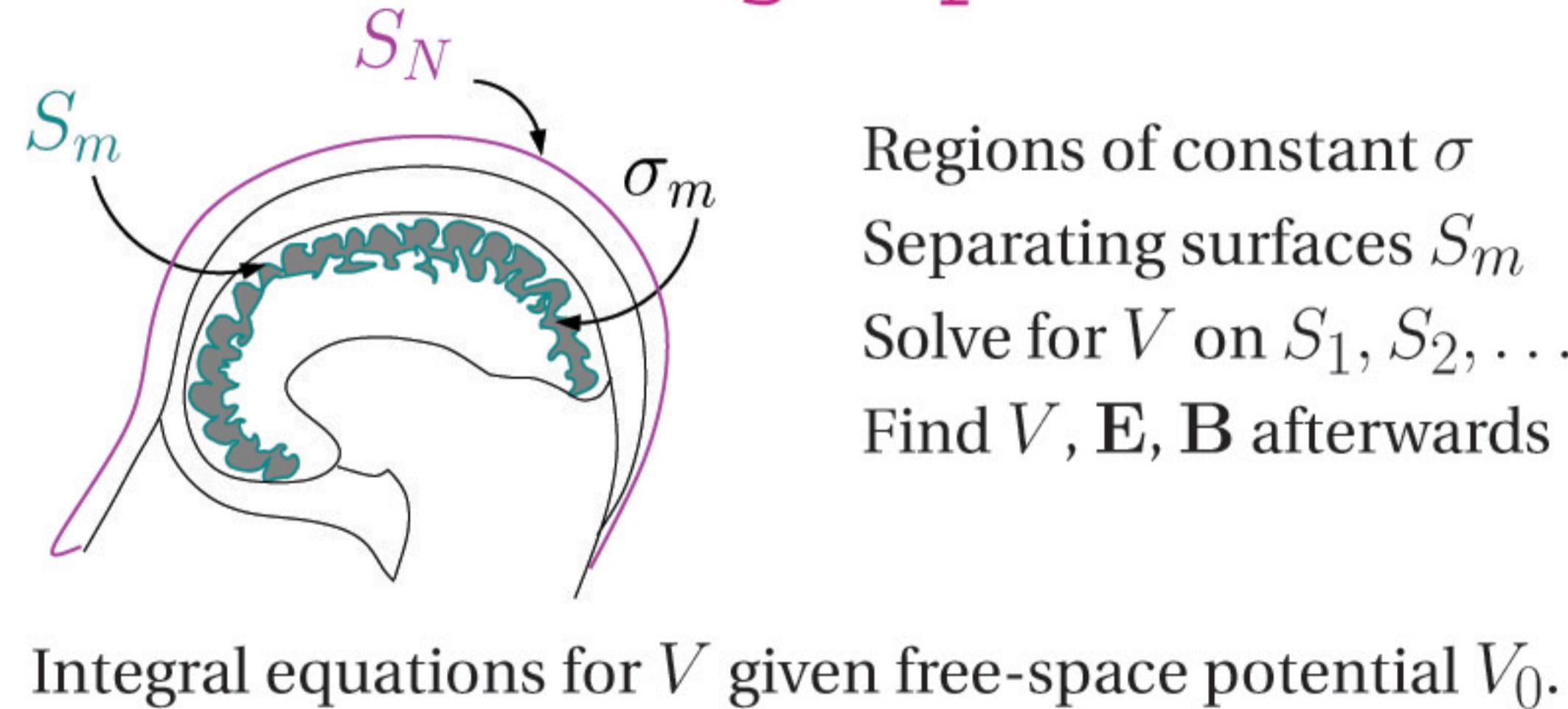


# The Fast Multipole Method for the Direct E/MEG Problem

## Abstract

Reconstructing neuronal activity from MEG and EEG measurements requires the accurate calculation of the electromagnetic field inside the head. The boundary element formulation of this problem leads to a dense linear system which is too large to be solved directly. We propose to accelerate the computations via the fast multipole method (FMM). This method approximates the electromagnetic interaction between surface elements by performing multipole expansions at coarse resolutions. It significantly reduces the computational burden of the matrix-vector products needed for the iterative solution of the linear system, and avoids the storage of its matrix. We present several experiments demonstrating the accuracy and performance of the single-level FMM.

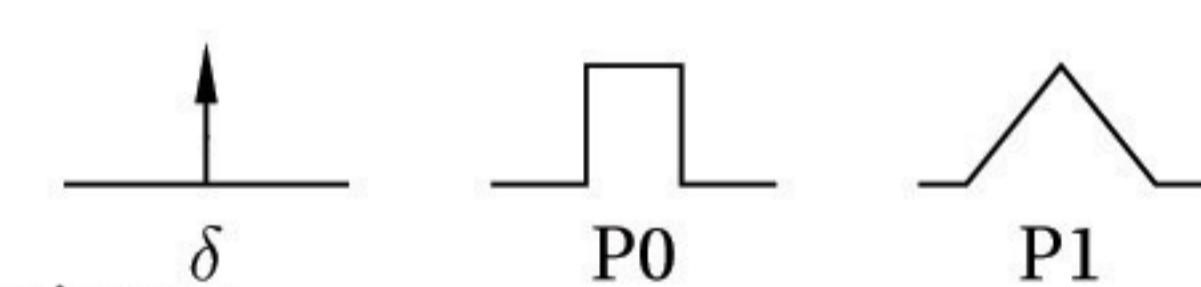
## Continuous integral problem



## Discrete problem

$$\text{Discretize: } V(\mathbf{r}) = \sum_{k=1}^N \sum_{i=1}^{P_k} v_{ik} \varphi_{ik}(\mathbf{r})$$

Basis functions  $\varphi, \psi$ :



Linear system of equations:

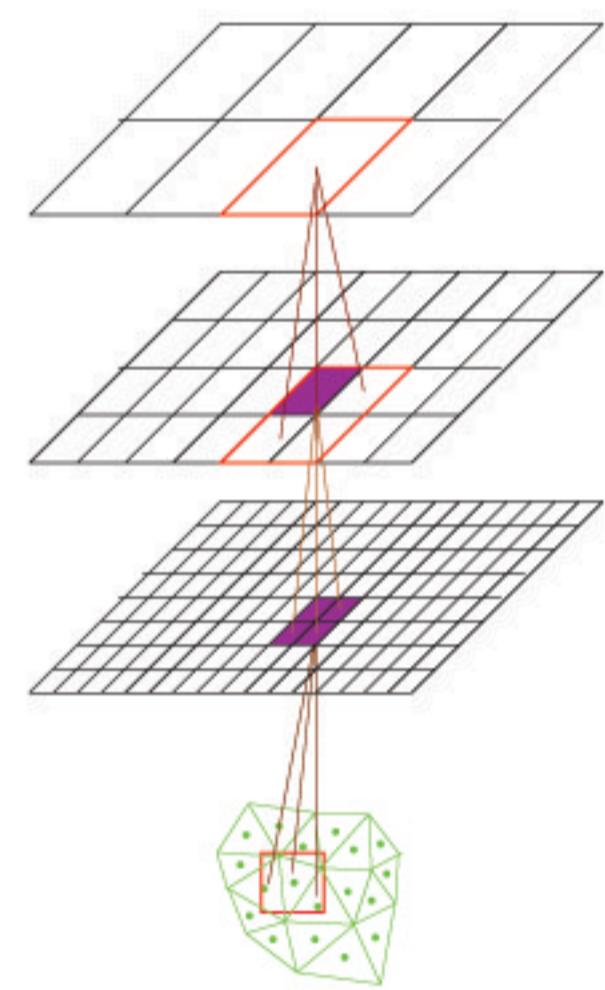
$$\langle \sigma_0 V_0, \psi_{im} \rangle = \frac{\sigma_m^+ + \sigma_m^-}{2} \sum_{j=1}^{P_m} v_{jm} \langle \varphi_{jm}, \psi_{im} \rangle + \sum_{k=1}^N \frac{\sigma_k^+ - \sigma_k^-}{4\pi} \sum_{j=1}^{P_k} v_{jk} \Gamma_{im,jk}$$

$$\Gamma_{im,jk} = \iint_{\substack{\mathbf{r} \in \text{supp } \psi_{im} \\ \mathbf{r}' \in \text{supp } \varphi_{jk}}} \nabla' \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} \cdot \mathbf{n}_{jk} \varphi_{jk}(\mathbf{r}') \psi_{im}(\mathbf{r}) d\mathbf{s}^2(\mathbf{r}', \mathbf{r})$$

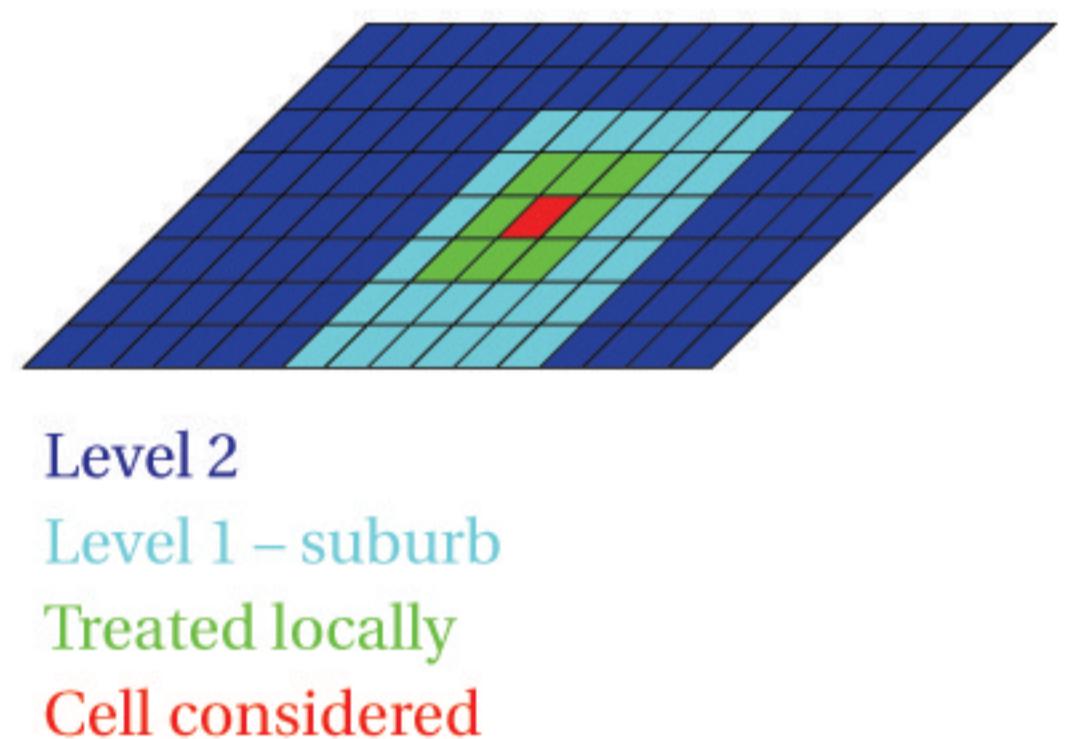
## Fast multipole method

Calculate approximately  $y_j = \sum_i \Gamma_{i,j} v_i$  for all  $j$ , in  $O(P \log P)$  time.

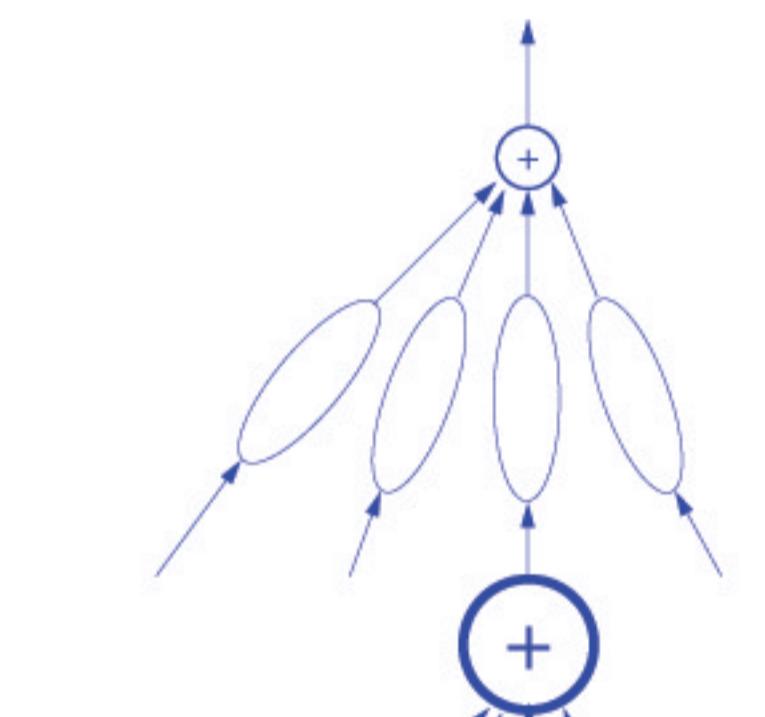
### 1. Build tree



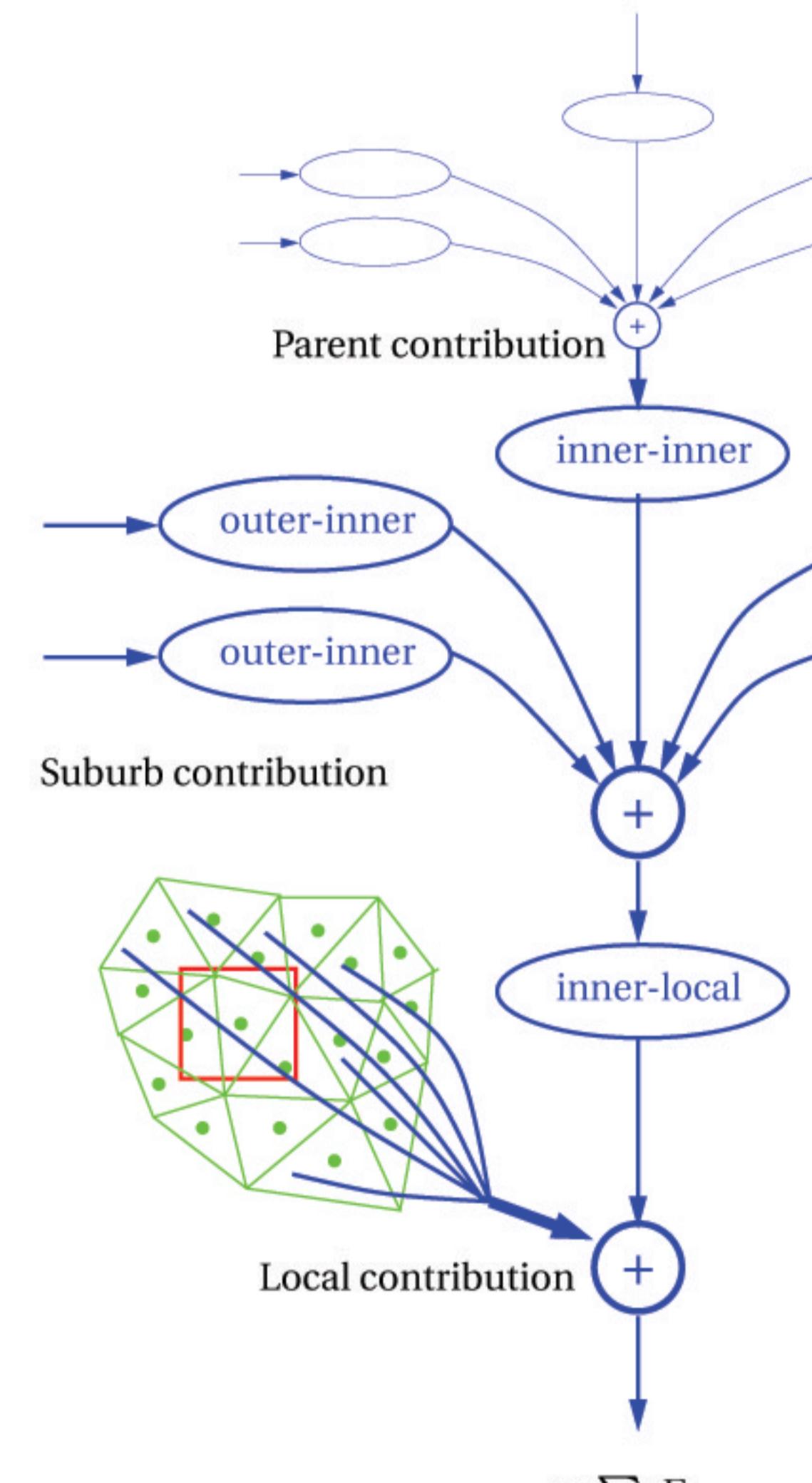
### The levels involved



### 2. Sweep-up $\rightarrow$ outer fields



### 3. Sweep-down $\rightarrow$ result



## Solution methods

### Direct

e.g. LU decomposition.  
 Complexity  $O(P^3)$ , memory  $O(P^2)$

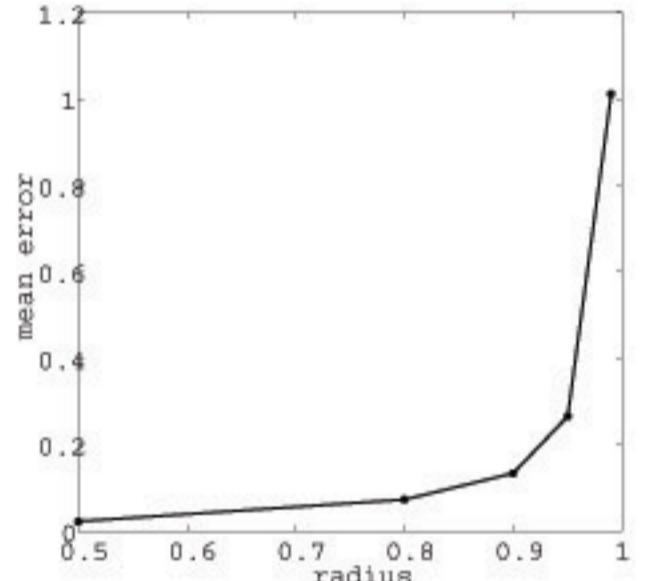
### Iterative

e.g. GMRES, uses products  $\Gamma \mathbf{v}$ .  
 Complexity  $O(MP^2)$ , memory  $O(P)$

Number of elements  $P$ , number of iterations  $M$ .

## Necessity of fine meshes

Dipole approaches surface  
 Error increases  
 Sources in cortex  
 $\Rightarrow 10^5 \sim 10^6$  triangles

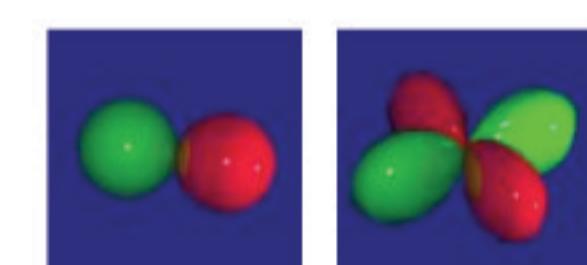


## Multipole expansion

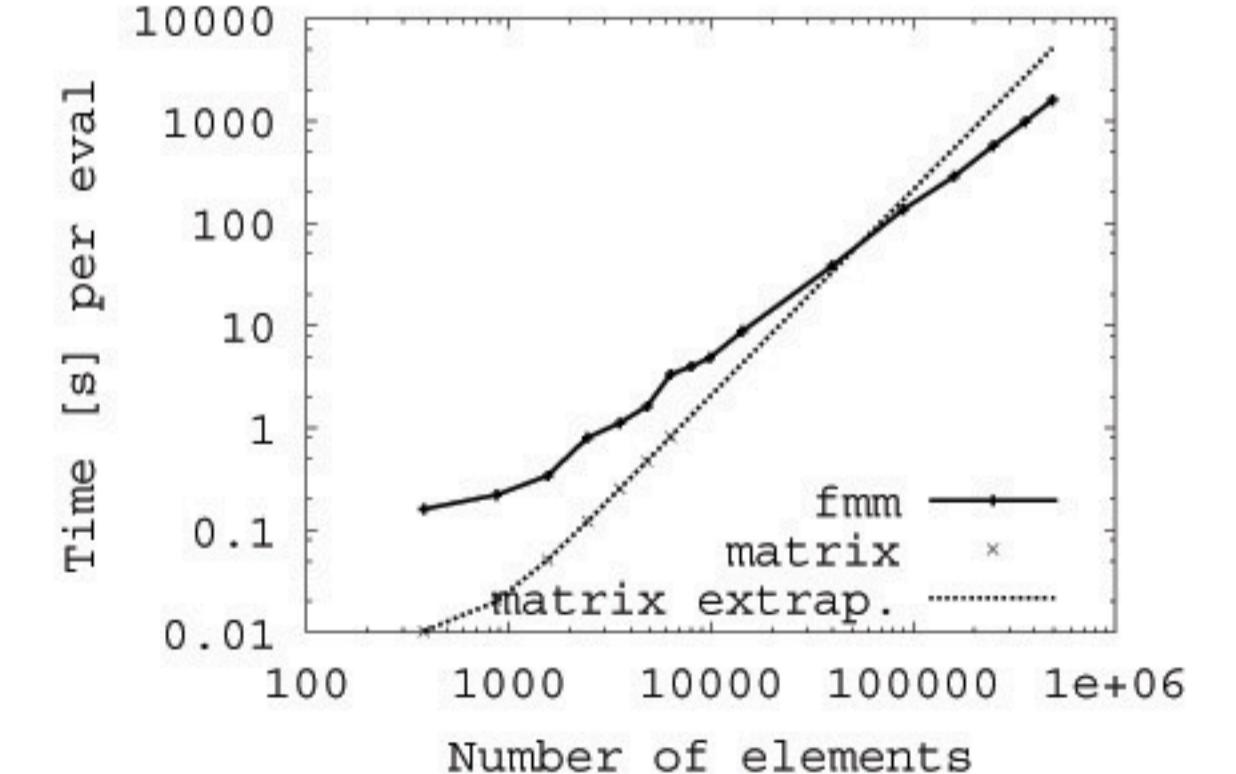
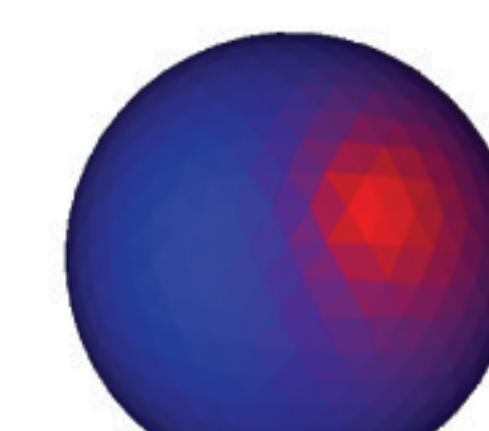
$$\nabla' \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = - \sum_{n=0}^{\infty} \sum_{m=-n}^{\infty} \nabla' I_n^{-m}(\mathbf{M}_p - \mathbf{r}') O_n^m(\mathbf{r} - \mathbf{M}_p)$$

Spherical harmonics

$$I_n^{-m}, O_n^m:$$



## Results



Single-level FMM,  $O(P^{4/3})$ , faster for  $P \gtrsim 70000$  triangles.