

# SOME RECENT RESULTS ON THE PROJECTIVE EVOLUTION OF 2-D CURVES

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## ABSTRACT

In this paper, we begin to explore the evolution of curves of the projective plane according to a family of intrinsic equations generalizing a “projective heat equation” [1]. This is motivated by previous work for the Euclidean [2, 3, 4] and the affine case [5, 6, 7, 8] as well as by applications in the perception of two-dimensional shapes. We establish the projective arclength evolution and the projective curvature evolution. Among this family of equations, we point out the ones preserving an important property of the Euclidean and affine heat equations that was not preserved in the projective case: a curve with constant curvature should remain such a curve during its evolution.

## 1. INTRODUCTION

Let  $\mathbf{B}(p, t) : R \times R \rightarrow \mathcal{P}^2$  be a family of smooth curves embedded in the real projective plane  $\mathcal{P}^2$ .  $t$  represents the time or the scale and  $p$  parameterizes the curve. Using standard results of projective differential geometry [9], we change  $\mathbf{B}(p, t)$  by a scale factor  $\lambda(p, t)$  and reparameterize the curves with their projective arclength  $\sigma(p, t)$  in such a way that the vectors  $\mathbf{A} = \lambda\mathbf{B}$ , considered as functions of  $\sigma$ , now satisfy the projective Frenet equations:

$$\begin{aligned} \frac{d\mathbf{A}}{d\sigma} &= \mathbf{A}^{(1)} \\ \frac{d\mathbf{A}^{(1)}}{d\sigma} &= -k\mathbf{A} + \mathbf{A}^{(2)} \\ \frac{d\mathbf{A}^{(2)}}{d\sigma} &= -\mathbf{A} - k\mathbf{A}^{(1)} \end{aligned} \quad (1)$$

$k$  is the projective curvature, the point  $\mathbf{A}^{(1)}$  is on the tangent to the curve in  $\mathbf{A}$  and the line  $\langle \mathbf{A}, \mathbf{A}^{(2)} \rangle$  is the projective normal. Functions  $k$  and  $\sigma$  are invariant under the action of the projective group  $PLG(2)$  and characterize the curve up to a projective transformation. Moreover, vectors  $\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}$  satisfy the condition:

$$|\mathbf{A}\mathbf{A}^{(1)}\mathbf{A}^{(2)}| = 1 \quad (2)$$

Note also that vectors  $\mathbf{A}$  and  $\mathbf{B}$  represent the same projective point.

Given a smooth curve  $\mathbf{B}_0(p)$ , trying to extend what had been done in the Euclidean [2, 3, 4] and the affine case

[5, 6, 7, 8], we previously studied [1] the evolution equation

$$\mathbf{A}_t = \mathbf{A}_{\sigma\sigma} \quad (3)$$

with initial condition  $\mathbf{A}(p, 0) = \mathbf{B}_0(p)$  in which the partial derivative with respect to time is taken at constant  $p$ . Hopelessly, while circles evolving according to the Euclidean heat equation remain circles, while the conics, which are the curves with constant affine curvatures, remain conics when they follow the affine heat equation, we noticed that equation (3) does not preserve the property of having a constant projective curvature. The motivation of this paper is then to find an intrinsic evolution equation close to equation (3) preserving this property. This could be one step further toward a projective scale-space.

## 2. CURVES WITH CONSTANT PROJECTIVE CURVATURE

Let us sketch out a way to determine the plane curves with constant projective curvature. For more detail, please refer to [10].

From the Frenet formulas (1), we get the useful relation

$$\mathbf{A}_{\sigma^3} = -2k\mathbf{A}^{(1)} - (1 + k_\sigma)\mathbf{A} \quad (4)$$

In this section, we don't deal with the evolution of curves anymore but only with a smooth curve  $\mathbf{A}(p)$ . If  $\mathbf{A}$  has a constant projective curvature, that is if  $k(p) = k_0$ , equation (4) becomes

$$\mathbf{A}_{\sigma^3} + 2k_0\mathbf{A}_\sigma + \mathbf{A} = 0 \quad (5)$$

We are thus led to consider the scalar equation

$$\theta''' + 2k_0\theta' + \theta = 0$$

If we write  $\theta = e^{r\sigma}$  we obtain the characteristic equation  $r^3 + 2k_0r + 1 = 0$ . Depending on its roots, we have the following cases (note we can't have three equal roots because their sum is equal to zero and their product is  $-1$ ):

1. *Three distinct real roots*  $r_1, r_2, r_3$ : the solutions of (5) are

$$\mathbf{A} = \mathbf{C}_1 e^{r_1\sigma} + \mathbf{C}_2 e^{r_2\sigma} + \mathbf{C}_3 e^{r_3\sigma}$$

where  $\mathbf{C}_i$  are fixed points. In a certain coordinate system:

$$\mathbf{A}^T = [x, y, z] = [e^{r_1\sigma}, e^{r_2\sigma}, e^{r_3\sigma}] \quad (6)$$

which is the homogeneous equation of the curve  $y = x^m$  (it can be proved that  $m$  can take any real value except  $2, \frac{1}{2}$  and  $-1$ )

2. One real root  $r_1$  and two complex conjugate roots  $r_2 \pm ir_3$ . In this case:

$$\mathbf{A}^T = [x, y, z] = [e^{r_1\sigma}, e^{r_2\sigma} \cos r_3\sigma, e^{r_2\sigma} \sin r_3\sigma] \quad (7)$$

or, in polar coordinates in the map  $[\frac{y}{x}, \frac{z}{x}]$ :

$$\rho = e^{(r_2-r_1)\sigma} \quad \theta = r_3\sigma$$

3. Two equal roots  $r_2 = r_3$ . There

$$\mathbf{A}^T = [x, y, z] = [e^{r_1\sigma}, e^{r_2\sigma}, \sigma e^{r_2\sigma}] \quad (8)$$

which leads to  $y = e^{(r_2-r_1)x}$  in the map  $[\frac{z}{y}, \frac{x}{y}]$ . Up to an homography, it is the exponential  $y = e^x$ .

This can be summarized as follows:

**Proposition 1** *The curves with constant projective curvature fall in three categories*

1. The general parabola:  $y = x^m, m \notin \{2, \frac{1}{2}, -1\}$
2. The logarithmic spiral:  $\rho = e^{m\theta}, m \neq 0$
3. The exponential:  $y = e^x$

### 3. EXTENDING THE PROJECTIVE HEAT EQUATION

When we study the evolution of curves with constant projective curvature under equation (3), we discover that their projective curvature does not remain constant as a function of  $p$ . Due to lack of space, we don't give here a direct proof of this. Anyway, this result is a consequence of proposition 4.

Noticing that equation (3) can be rewritten  $\mathbf{A}_t = -k\mathbf{A} + \mathbf{A}^{(2)}$  using equation (1), we will consider the following evolution laws:

$$\mathbf{A}_t = \alpha\mathbf{A} + \beta\mathbf{A}^{(1)} + \gamma\mathbf{A}^{(2)} \quad (9)$$

where  $\alpha, \beta, \gamma$  are some differential projective invariants of the curves  $\mathbf{A}(p, t)$ .

The projective arclength  $\sigma$  and of the projective curvature  $k$  characterizing a curve up to an homography, it is natural to study their evolution laws. Actually, it is usual to study  $g = \frac{\partial\sigma}{\partial p}$  instead of  $\sigma$ .

Let us first establish some preliminary properties. Using the fact that the independent variables  $p$  and  $t$  verify

$$\frac{\partial^2}{\partial t \partial p} = \frac{\partial^2}{\partial p \partial t}$$

it is quite immediate to show that the Lie bracket  $[\frac{\partial}{\partial t}, \frac{\partial}{\partial \sigma}]$  equals:

$$[\frac{\partial}{\partial t}, \frac{\partial}{\partial \sigma}] = \frac{\partial^2}{\partial t \partial \sigma} - \frac{\partial^2}{\partial \sigma \partial t} = -\frac{g_t}{g} \frac{\partial}{\partial \sigma} \quad (10)$$

Applying this formula twice more, we obtain the expressions

$$\frac{\partial^3}{\partial t \partial \sigma^2} = -[\frac{g_t}{g}]_\sigma \frac{\partial}{\partial \sigma} - 2\frac{g_t}{g} \frac{\partial^2}{\partial \sigma^2} + \frac{\partial^3}{\partial \sigma^2 \partial t} \quad (11)$$

and

$$\frac{\partial^4}{\partial t \partial \sigma^3} = -[\frac{g_t}{g}]_{\sigma^2} \frac{\partial}{\partial \sigma} - 3[\frac{g_t}{g}]_\sigma \frac{\partial^2}{\partial \sigma^2} - 3\frac{g_t}{g} \frac{\partial^3}{\partial \sigma^3} + \frac{\partial^4}{\partial \sigma^3 \partial t} \quad (12)$$

that we will need further.

#### 3.1. Evolution of the projective arclength

We use equation (2). Taking the partial derivative of the two members with respect to  $t$  at constant  $p$ , we obtain the desired equation.

**Proposition 2** *The evolution law of the projective arclength of a curve following equation (9) is given by*

$$\frac{g_t}{g} = \alpha + \beta_\sigma + \frac{\gamma_{\sigma^2} - k\gamma}{3} \quad (13)$$

**Proof:** We have

$$|\mathbf{A}\mathbf{A}^{(1)}\mathbf{A}^{(2)}|_t = |\mathbf{A}_t\mathbf{A}^{(1)}\mathbf{A}^{(2)}| + |\mathbf{A}\mathbf{A}_t^{(1)}\mathbf{A}^{(2)}| + |\mathbf{A}\mathbf{A}^{(1)}\mathbf{A}_t^{(2)}| \quad (14)$$

From equations (9) and (1), the first determinant of the right hand side member equals  $\alpha$ . Using also equation (10), we get

$$\mathbf{A}_t^{(1)} = \mathbf{A}_{t\sigma} = -\frac{g_t}{g}\mathbf{A}^{(1)} + \frac{\partial\mathbf{A}_t}{\partial\sigma} \quad (15)$$

from which we finally obtain the value of  $|\mathbf{A}\mathbf{A}_t^{(1)}\mathbf{A}^{(2)}|$  in (14). In a similar way, we write

$$\mathbf{A}_t^{(2)} = \frac{\partial k\mathbf{A} + \mathbf{A}_{\sigma\sigma}}{\partial t} = k_t\mathbf{A} + k\mathbf{A}_{\sigma\sigma} + \mathbf{A}_{t\sigma\sigma} \quad (16)$$

whose last term is obtained from equation (11). Thus the value of the last determinant in (14) and finally equation (13)  $\square$

#### 3.2. Evolution of the projective curvature

**Proposition 3** *The evolution law of the projective curvature of a curve following equation (9) is given by*

$$k_t = -\alpha_{\sigma^2} + \frac{3}{2}\gamma_\sigma + \frac{\gamma_{\sigma^4}}{6} + k(\frac{2}{3}\gamma_{\sigma^2} - 2\alpha) + k_\sigma(\beta + \frac{7}{6}\gamma_\sigma) + \frac{\gamma}{3}(k_{\sigma^2} + 2k^2) \quad (17)$$

**Proof:** Let us compute  $\frac{\partial}{\partial t}\mathbf{A}_{\sigma^3}$  in two different ways:

1. Using equation (4), we have

$$\begin{aligned} \frac{\partial\mathbf{A}_{\sigma^3}}{\partial t} &= \frac{\partial}{\partial t}(-(1+k_\sigma)\mathbf{A} - 2k\mathbf{A}^{(1)}) \\ &= -\frac{\partial k_\sigma}{\partial t}\mathbf{A} - (1+k_\sigma)\mathbf{A}_t - 2k_t\mathbf{A}^{(1)} - 2k\mathbf{A}_t^{(1)} \end{aligned} \quad (18)$$

where we know  $\mathbf{A}_t$  and  $\mathbf{A}_t^{(1)}$  from (9) and (15).

2. On the other hand, we can use equation (12)

$$\frac{\partial \mathbf{A}_{\sigma^3}}{\partial t} = -\left[\frac{g_t}{g}\right]_{\sigma^2} \mathbf{A}_{\sigma} - 3\left[\frac{g_t}{g}\right]_{\sigma} \mathbf{A}_{\sigma^2} - 3\frac{g_t}{g} \mathbf{A}_{\sigma^3} + \frac{\partial^3 \mathbf{A}_t}{\partial \sigma^3} \quad (19)$$

where all the terms of the right hand side member are known from (13) and (1), except the last one which can be computed from equation (9) and the Frenet formulas.

Expressing (18) and (19) as functions of  $(\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$  and equating the coefficient of  $\mathbf{A}^{(1)}$  in both, we get (17)  $\square$

#### 4. PRESERVING CURVES WITH CONSTANT PROJECTIVE CURVATURE

We want to choose  $(\alpha, \beta, \gamma)$  such that an initial curve with constant projective curvature evolves while keeping this property. Here is our main result:

**Proposition 4** *1. Curves with constant projective curvature following evolution equation (9) remain curves with constant curvature if and only if differential projective invariants  $\alpha, \beta, \gamma$  verify*

$$k\gamma = 3\alpha \quad (20)$$

*2. In that case, these curves keep their initial curvature ( $k_t = 0$ ), hence remaining the same curve up to an homography. Projective arclength and curvature of any curve evolve the following way*

$$\begin{cases} \frac{g_t}{g} = \beta_{\sigma} + \frac{\gamma_{\sigma^2}}{3} \\ k_t = \frac{3}{2}\gamma_{\sigma} + \frac{\gamma_{\sigma^4}}{6} + \frac{k}{3}\gamma_{\sigma^2} + k_{\sigma}\left(\beta + \frac{1}{2}\gamma_{\sigma}\right) \end{cases} \quad (21)$$

**Proof:** We have  $k(p, t) = k(t)$ . Let  $r_i(t)$  be the solutions of

$$r^3 + 2k(t)r + 1 = 0 \quad (22)$$

There is an homography  $H(t)$  such that

$$\mathbf{A}(p, t) = H(t) \cdot [e^{r_1 \sigma}, e^{r_2 \sigma}, e^{r_3 \sigma}]^T$$

(or one of the other two expressions (7) or (8)). It can be shown that the coefficient of the term in  $e^{r_1 \sigma}$  of the first coordinate of the evolution law (9) has to vanish, that is

$$(H_{11})_t + H_{11}(r_1 \sigma)_t = H_{11}(\alpha + \beta r_1 + \gamma(r_1^2 + k))$$

hence

$$\sigma_t = \frac{1}{r_1}[(\alpha + \beta r_1 + \gamma(r_1^2 + k)) - \frac{(H_{11})_t}{H_{11}} - (r_1)_t \sigma] \quad (23)$$

The derivative of this equation with respect to  $\sigma$  is

$$\frac{\partial \sigma_t}{\partial \sigma} = \frac{1}{r_1}[(\alpha_{\sigma} + \beta_{\sigma} r_1 + \gamma_{\sigma}(r_1^2 + k)) - (r_1)_t] \quad (24)$$

equation whose left hand side member is given by (10)

$$\frac{\partial \sigma_t}{\partial \sigma} = \frac{\partial \sigma_{\sigma}}{\partial t} + \frac{g_t}{g} \frac{\partial \sigma}{\partial \sigma} = \frac{g_t}{g} \quad (25)$$

and whose last term of the right hand side member is obtained deriving (22) with respect to  $t$

$$(r_1)_t = \frac{-2r_1}{3r_1^2 + 2k} k_t \quad (26)$$

Using equations (13), (17), (25), (22) and (26) in (24), it comes

$$\gamma_{\sigma} r_1^4 + \alpha_{\sigma} r_1^2 + \alpha_{\sigma} + 3(3\alpha - k\gamma) = 0 \quad (27)$$

It can be shown that the only way to satisfy (27) and similar equations for  $r_2$  and  $r_3$  (or for their real and imaginary parts) is to verify

$$\begin{cases} \alpha_{\sigma} = 0 \\ \gamma_{\sigma} = 0 \\ 3\alpha - k\gamma = 0 \end{cases} \quad (28)$$

We know from classical results in the theory of differential invariants (see [11]) that every projective differential invariant is a function  $I(k, k_{\sigma}, \dots, k_{\sigma^n})$  for some  $n$ . Thus, the first two conditions of (28) are verified (remember  $k = k(t)$ ). Finally, only equation (20) remains. Moreover, this condition implies from (17) that  $k_t = 0$ . Hence the first part of point 2 in proposition 4 and the fact that the necessary condition (20) is sufficient. Equations (21) come directly from (13) and (17) using (20)  $\square$

Note that the projective heat equation doesn't verify condition (20), hence a proof of what is said at the beginning of section 3.

Note also that any  $\beta$  may be chosen. This should not be surprising because  $\beta$  governs the evolution along the tangent. Whatever its value, the same family of curves is obtained up to a reparameterization in  $p$ .

A simple case is the evolution law  $\mathbf{A}_t = k\mathbf{A} + 3\mathbf{A}^{(2)}$ . For this equation, (21) becomes  $g_t = k_t = 0$ . Hence, any curve evolves remaining the initial one up to an homography. Figure 1 shows the evolution of a logarithmic spiral, rotating and shrinking. Figure 2 shows what a general parabola becomes, tending to a line. See figure 3 for the exponential curve. Nevertheless, this equation doesn't smooth curves, making it somewhat useless.

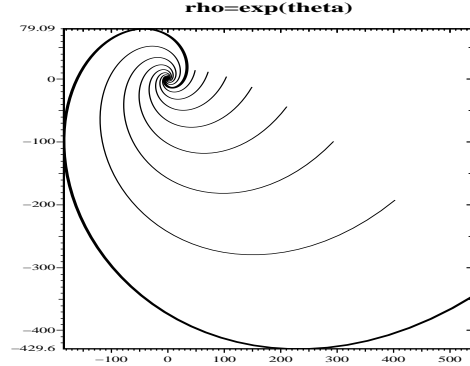


Figure 1: A logarithmic spiral evolving according to  $\mathbf{A}_t = k\mathbf{A} + 3\mathbf{A}^{(2)}$ . In bold, the initial curve

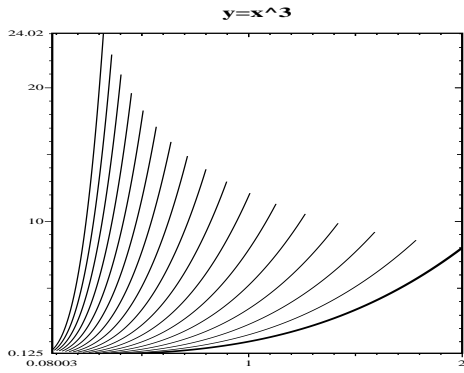


Figure 2: A general parabola evolving according to  $\mathbf{A}_t = k\mathbf{A} + 3\mathbf{A}^{(2)}$ . In bold, the initial curve

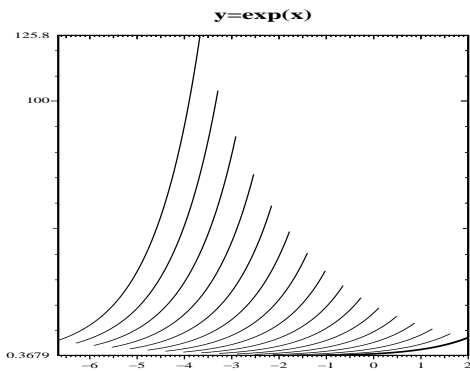


Figure 3: The exponential curve evolving according to  $\mathbf{A}_t = k\mathbf{A} + 3\mathbf{A}^{(2)}$ . In bold, the initial curve

The general case seems more promising. We currently investigate what kind of values of  $\gamma$  lead to interesting evolutions of the projective arclength and curvature. It appears that system (21) could have smoothing properties for good choices of  $\gamma$ .

## 5. CONCLUDING REMARKS

This work sets the basis for the study of projective invariant evolutions of plane curves preserving the property of having a constant projective curvature. We establish the evolution laws of the projective arclength and curvature. There remains to find out which particular evolutions lead to interesting properties and to see if it is possible, as in the Euclidean and affine cases, to define a projective scale-space.

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