On projective plane curve evolution

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Abstract

In this paper, we investigate the evolution of curves of the projective plane according to a family of projective invariant intrinsic equations. This is motivated by previous work for the Euclidean [11, 12, 14] and the affine cases [21, 22, 3, 2] as well as by applications in the perception of two-dimensional shapes. We establish the evolution laws for the projective arclength and curvature. Among this family of equations, we define a "projective heat equation" [7] and establish the link with the projective evolution of curves in $\mathbf{R}^2$.

Keywords: multi-scale analysis, partial differential equations, projective geometry

1 Introduction

The use of partial differential equations and curve or surface evolution theory in image analysis became a major research topic in the past years (see [18]) leading to applications in image de-noising and de-blurring [19], in selective smoothing and edge detection [1, 17], in contrast enhancement [20], in shape segmentation [5]. Recently, applications were found in problems usually addressed by the computer vision community: intrinsic flows [14, 21] hold very good geometric smoothing properties and allow the computation of local differential invariants [9]. Motivated by the importance of projective geometry in computer vision, we found it natural to extend the Euclidean [14] and affine [21] cases to the projective one.

2 Geometric flows

Let $\mathcal{L}$ be a Lie group operating on some objects. A quantity q depending on these objects is called an *invariant* of $\mathcal{L}$ if, whenever a transformation $L \in \mathcal{L}$ changes q into q', we have $q' = \alpha(L)q$, where α is a function of L alone, i.e. does not depend on the object which is transformed. If $\alpha \equiv 1$, then q is called an *absolute invariant*.

Differential invariants are special invariants based on local transformations (see [13]).

Let $\mathcal{C} : \mathbf{R} \to \mathbf{R}^2$ be a plane curve of parameter p. The first and the second differential invariants for the Euclidean group $\{m \mapsto Rm + T \mid R \text{ rotation}, \text{T translation}\}$ are the well known Euclidean arclength v and curvature κ defined by:

$$\left\{\begin{array}{lcl} \frac{\partial v}{\partial p} & = & \|\frac{\partial \mathcal{C}}{\partial p}\| \\ \kappa & = & \|\frac{\partial^2 \mathcal{C}}{\partial v^2}\| \end{array}\right. \tag{1}$$

which are preserved by rotations and translations.

The corresponding invariants for the group of proper affine motions $\{m \mapsto Am+B \mid [A]>0, B \in \mathbf{R}^2\}$ are the affine arclength s and curvature μ defined by:

$$\left\{\begin{array}{lcl} \frac{\partial s}{\partial p} & = & [\frac{\partial \mathcal{C}}{\partial p}, \frac{\partial^2 \mathcal{C}}{\partial p^2}]^{1/3} \\ \mu & = & [\frac{\partial^2 \mathcal{C}}{\partial s^2}, \frac{\partial^3 \mathcal{C}}{\partial s^3}] \end{array}\right. \tag{2}$$

which are invariants for affine proper motions, and absolute invariants for special affine motions ($\{m \mapsto Am + B \mid [A] = 1, B \in \mathbf{R}^2\}$) .

Circles (and straight lines) are the only curves with constant Euclidean curvature. In the affine case, constant affine curvature is obtained for the conics ($\mu = 0$ for a parabola, $\mu > 0$ for an ellipse and $\mu < 0$ for an hyperbola).

Given an initial plane curve $\mathcal{C}_0(p) : \mathbf{R} \to \mathbf{R}^2$, the associated geometric flow (see [16]) is the family of curves $\mathcal{C}(p,t) : \mathbf{R} \times [0, \tau) \to \mathbf{R}^2$ evolving according to the following law:

$$\left\{\begin{array}{lcl} \frac{\partial \mathcal{C}(p,t)}{\partial t} & = & \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2} \\ \mathcal{C}(p,0) & = & \mathcal{C}_0(p) \end{array}\right. \tag{3}$$

where r is the group arclength (v for the Euclidean geometric flow, s for the affine one). Contrary to the classical heat flow $\mathcal{C}_t = \mathcal{C}_{pp}$, these flows are intrinsic (i.e. don't depend on the parameterization p of the initial curve). They are invariant for the considered Lie group. Their "smoothing" properties may be summarized as follow ([14, 21]): closed curves evolve toward a convex one and then disappear shrinking toward a circle point (Euclidean case) or an ellipse point (affine case).

For a given group, a plane curve is defined up to a group transformation by its group arclength and curvature. Hence, it is natural to study these flows through the evolution of the arclength and curvature. With $g_e = \frac{dv}{dp}$ and $g_a = \frac{ds}{dp}$, we have:

$$\left\{\begin{array}{lcl} \frac{\partial g_e}{\partial t} & = & -g_e \kappa^2 \\ \frac{\partial \kappa}{\partial t} & = & -\kappa^3 - \frac{\partial^2 \kappa}{\partial v^2} \end{array}\right. \text{ and } \left\{\begin{array}{lcl} \frac{\partial g_a}{\partial t} & = & -2 g_a \mu / 3 \\ \frac{\partial \mu}{\partial t} & = & \frac{4}{3}\mu^2 + \frac{1}{3}\frac{\partial^2 \mu}{\partial s^2} \end{array}\right. \tag{4}$$

3 Projective geometry

Like in equations (1) and (2), it is possible to define the projective arclength and curvature of a plane curve in $\mathbf{R}^2$. However, this leads to too complex expressions. The idea is to embed such a curve in the real projective plane $\mathcal{P}^{\in}$. One can see $\mathcal{P}^{\in}$ as the set of the lines of $\mathbf{R}^3$ going through the origin. An element of $\mathcal{P}^{\in}$ is represented by its homogeneous coordinates (x, y, z) where (x, y, z) and $(\lambda x, \lambda y, \lambda z), (\lambda \neq 0)$ are different coordinate vectors of the same projective point.

Let $\mathbf{B}(p) : \mathbf{R} \to \mathcal{P}^{\in}$ be a smooth curve of the projective plane. Using standard results of projective differential geometry [4], we change $\mathbf{B}(p)$ by a scale factor $\lambda(p)$ and characterize its projective arclength σ and curvature k introducing the Cartan point $\mathbf{A} = \lambda\mathbf{B}$, and the Cartan frame $(\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$ which satisfy the projective Frenet equations:

$$\begin{aligned} \frac{d\mathbf{A}}{d\sigma} &= \mathbf{A}^{(1)} \\ \frac{d\mathbf{A}^{(1)}}{d\sigma} &= -k\mathbf{A} + \mathbf{A}^{(2)} \\ \frac{d\mathbf{A}^{(2)}}{d\sigma} &= -\mathbf{A} - k\mathbf{A}^{(1)} \end{aligned} \tag{5}$$

and the condition:

$$|\mathbf{A}\mathbf{A}^{(1)}\mathbf{A}^{(2)}| = 1 \tag{6}$$

Note that $\mathbf{B}$ and $\mathbf{A}$ are different coordinate vectors of the same projective point. The point $\mathbf{A}^{(1)}$ is on the tangent to the curve in $\mathbf{A}$ and the line $\langle \mathbf{A}, \mathbf{A}^{(2)} \rangle$ is the projective normal. Functions k and σ are invariant under the action of the projective group and characterize the curve up to a projective transformation.

The plane curves with a constant projective curvature are (see [10]):

- If $k = k_0 = -3/32^{1/3}$: the exponential ($y = e^x$)
- If $k < k_0$: the general parabola ($y = x^m, m \notin \{2, \frac{1}{2}, -1\}$)
- If $k > k_0$: the logarithmic spiral ($\rho = e^{m\theta}, m \neq 0$)

4 Projective invariant intrinsic flows

The law $\mathbf{A}_t = \mathbf{A}_{\sigma\sigma}$ investigated in [7] could be thought as a natural extension of the Euclidean and affine cases. Yet, this law raises some contradictions. For instance, according to the expression of k_t in [7], curves with a constant initial curvature should evolve keeping a constant curvature. Actually, it's not the case (see [10]).

The reason why it is so is that the Cartan point $\mathbf{A}(p,t)$ is some particular representant for the projective point $\mathbf{B}(p,t)$ and depends on the curve and its spatial derivatives at (p,t). As a result, one can't expect an arbitrary differential equation $\{\mathbf{A}(p,0) = \mathbf{A}_0(p); \mathbf{A}_t = f(p,t)\}$ to be such that $\mathbf{A}(p,t)$ will still be the Cartan point of the curve at time $t > 0$.

This leads us to consider the evolution law

$$\begin{cases} \mathbf{A}(p,0) = \mathbf{A}_0(p) \quad (\mathbf{A}_0 \text{ Cartan point of the initial curve}) \\ \mathbf{A}_t(p,t) = \alpha\mathbf{A} + \beta\mathbf{A}^{(1)} + \gamma\mathbf{A}^{(2)} \end{cases} \tag{7}$$

where $f(p,t)$ has been decomposed on the Cartan frame, and to find out which conditions on (α, β, γ) will assure that $\mathbf{A}(p,t)$ remains the Cartan point.

Another way to see this is to consider the surface $\mathcal{S} = \{\mathbf{A}(\surd, \sqcup)\}$ of $\mathbf{R}^3$. The reason why this is a well-defined surface is because there is no scale factor on $\mathbf{A}$ even though it represents a projective point of $\mathcal{P}^{\in}$. Now, in order for (7) to be a well-defined PDE on $\mathcal{S}$,the vector $\mathbf{A}_t$ has to belong to the tangent plane $T_{\mathcal{S}}$ at (p,t). The right hand side contains the vector $\mathbf{A}^{(1)}$ which belongs to $T_{\mathcal{S}}$ but the vector $\alpha\mathbf{A}+\gamma\mathbf{A}^{(2)}$ does not in general belong to $T_{\mathcal{S}}$ unless α and γ are dependent In fact the condition is even stronger since not only $\mathbf{A}_t$ must belong to $T_{\mathcal{S}}$ but also, as stated above, $\mathbf{A}$ must remain a Cartan point.

We get the following result (see [8] for the proof):

Proposition 1 *The differential equation (7) has a meaning (i.e.* $\mathbf{A}(p,t)$ *is the Cartan point of the curve at time t) if and only if:*

$$\alpha = \frac{1}{3+k_\sigma} \quad [\quad -\frac{1}{3}k_{\sigma^3} - \frac{3}{2}k_{\sigma^2}\gamma_\sigma - k_\sigma(\frac{7}{3}k\gamma + \frac{17}{6}\gamma_{\sigma^2} + \beta_\sigma) - \frac{8}{3}k^2\gamma_\sigma \\ +k(\gamma - \frac{5}{3}\gamma_{\sigma^3}) + \gamma_{\sigma^2}/2 - \gamma_{\sigma^5}/6 \quad] \tag{8}$$

In this case, the projective arclength and curvature evolve according to:

$$\frac{g_t}{g} = \alpha + \beta_\sigma - \frac{1}{3}(k\gamma - \gamma_{\sigma^2}) \tag{9}$$

$$k_t = -\alpha_{\sigma^2} + \frac{3}{2}\gamma_\sigma + \frac{\gamma_{\sigma^4}}{6} + k(\frac{2}{3}\gamma_{\sigma^2} - 2\alpha) \\ +k_\sigma(\beta + \frac{7}{6}\gamma_\sigma) + \frac{\gamma}{3}(k_{\sigma^2} + 2k^2) \tag{10}$$

where $g = \frac{d\sigma}{dp}$.

Note that $\mathbf{A}_t = \mathbf{A}_{\sigma\sigma}$ is the case $(\alpha,\beta,\gamma) = (-k,0,1)$, thus doesn't satisfy condition (8), hence the previous contradictions.

Moreover, if β and γ are projective invariant intrinsic quantities, then α defined by equation (8) is a projective invariant intrinsic quantity too. Therefore, we get:

Corollary 1 *Let β and γ be some projective invariant intrinsic quantities, let α be defined by equation (8) . The differential equation (7) defines a projective invariant intrinsic flow. The evolution of the projective arclength and curvature of the curves is given by equations (9, 10).*

5 The projective "heat flow"

Among all the possible choices for (β,γ), it turns out that the simplest one $(0,1)$ is also the right one for a projective "heat flow" extending the Euclidean and affine cases. Some intuitive justification could be:

- $\beta\mathbf{A}^{(1)}$ is on the tangent in $\mathbf{A}$. Thus, the choice of β has no importance: changing β doesn't modify the family of curves obtained but only their parameterization p (see [21]).
- $(\beta,\gamma) = (0,1)$ are the components of $\mathbf{A}_{\sigma\sigma}$ on $(\mathbf{A}^{(1)},\mathbf{A}^{(2)})$. The induced α could be considered as a corrected component on $\mathbf{A}$.

However, the deep reason for this choice is that it gives the same flow as $\mathcal{C}_t = \mathcal{C}_{\sigma\sigma}$ in $\mathbf{R}^2$ (see next section). Consequently, we have from proposition 1 the following statement:

Proposition 2 *Let α be:*

$$\alpha = \frac{1}{9+3k_\sigma}(3k - 7kk_\sigma - k_{\sigma^3})$$

Let $\mathbf{B}_0(p)$ be a curve of $\mathcal{P}^\in$ and $\mathbf{A}_0(p)$ its Cartan points. We define its projective heat flow as the solution of:

$$\left\{ \begin{array}{lcl} \mathbf{A}(p,0) & = & \mathbf{A}_0(p) \\ \mathbf{A}_t(p,t) & = & \alpha\mathbf{A} + \mathbf{A}^{(2)} \end{array} \right. \tag{11}$$

Let $g = \frac{d\sigma}{dp}$. The projective arclength and curvature evolve according to:

$$\frac{g_t}{g} = \frac{-1}{9+3k_\sigma}(8kk_\sigma + k_{\sigma^3}) \tag{12}$$

$$k_t = \frac{2}{3}k^2 + \frac{1}{3}k_{\sigma^2} - 2\alpha k - \alpha_{\sigma^2} \tag{13}$$

6 Going back to $\mathbf{R}^2$

We prove in [8] that:

Proposition 3 *Given an initial curve in $\mathcal{P}^\in$, let $\mathbf{B}_0(p)$ be any coordinate vector of it.*

1. *The flow defined by*

$$\left\{ \begin{array}{lcl} \mathbf{B}(p,0) & = & \mathbf{B}_0(p) \\ \mathbf{B}_t(p,t) & = & \mathbf{B}_{\sigma\sigma} \end{array} \right. \tag{14}$$

   is intrinsic and doesn't depend on the choice of $\mathbf{B}_0$ (i.e. $\mathbf{B}_0(p)$ and $\phi(p)\mathbf{B}_0(p)$ give the same family of curves).

2. *This flow is the projective heat flow defined by equation (11) up to a parameterization of the curves.*

3. *Let λ be the Cartan scale factor ($\mathbf{A} = \lambda\mathbf{B}$). (σ, k, λ) define $\mathbf{B}$ up to a projective transformation. Their evolution is given by:*

$$\begin{array}{lcl} \frac{g_t}{g} & = & \frac{-1}{9+3k_\sigma}(8kk_\sigma + k_{\sigma^3} + 18\Lambda_{\sigma^2}) \\ k_t & = & \frac{2}{3}k^2 + \frac{1}{3}k_{\sigma^2} - 2Pk - P_{\sigma^2} - 2k_\sigma\Lambda_\sigma \\ \Lambda_t & = & \frac{-1}{9+3k_\sigma}[k_{\sigma^3} + 3k_\sigma(\Lambda_\sigma^2 - 3\Lambda_{\sigma^2}) + 4k(k_\sigma - 3) + 9(\Lambda_\sigma^2 - \Lambda_{\sigma^2})] \\ & & \text{where } \mathrm{g} = \frac{\mathrm{d}\sigma}{\mathrm{dp}},\ \Lambda = \log|\lambda|,\ \mathrm{P} = \Lambda_\sigma^2 - \Lambda_{\sigma^2} - \mathrm{k} + \Lambda_\mathrm{t} \end{array} \tag{15}$$

Let $\mathcal{C}_0(p) = (x_0, y_0)$ be a real plane curve, it is then easy to prove that:

Corollary 2 *The flow defined by $\{\mathcal{C}(p,0) = \mathcal{C}_0 \ ; \ \mathcal{C}_t = \mathcal{C}_{\sigma\sigma}\}$ is a projective invariant flow. It gives the same family of curves through the map $(\frac{x}{z}, \frac{y}{z})$ as the projective heat flow (11) with initial curve $(x_0, y_0, 1)$. Let $\mathcal{C}(p,t) = (x, y)$ and λ be the Cartan scale of $(x, y, 1)$, the evolution of the projective arclength and curvature of $\mathcal{C}$ is given by equations (15).*

This was already proved in [15], even though the argument in [16] about the relationship between different coordinate vectors is incorrect (see proposition 3 above)

7 Conclusion

We have established a link between the invariant projective flow defined in $\mathbf{R}^2$ [16, 15] and the one defined in $\mathcal{P}^{\in}$ [7]. We have defined the projective heat equation in three equivalent ways: $\mathbf{A}_t = \alpha\mathbf{A} + \mathbf{A}^{(2)}$ (α given by equation (8)) or $\mathbf{B}_t = \mathbf{B}_{\sigma\sigma}$ in $\mathcal{P}^{\in}$, and $\mathcal{C}_t = \mathcal{C}_{\sigma\sigma}$ in $\mathbf{R}^2$. As expected, the connection is not trivial but simple enough. The advantage of the definition in $\mathcal{P}^{\in}$ [7] which we have modified here to make it entirely correct is that: a) it does not depend on the particular coordinates used to represent $\mathcal{P}^{\in}$ and b) it has allowed us to establish the evolution of the projective arclength and curvature. There remains to see if it is possible to define a projective scale-space as in the Euclidean and affine cases. Of particular interest would be to compare our approach with the one developed by Dibos [6].

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