Learning with Inference for Discrete Graphical Models

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Introduction

Conditional Random Fields (CRFs)

- Ubiquitous in computer vision
  - segmentation — stereo matching
    optical flow — image restoration
    image completion — object detection/localization
    ...

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- Really powerful formulation

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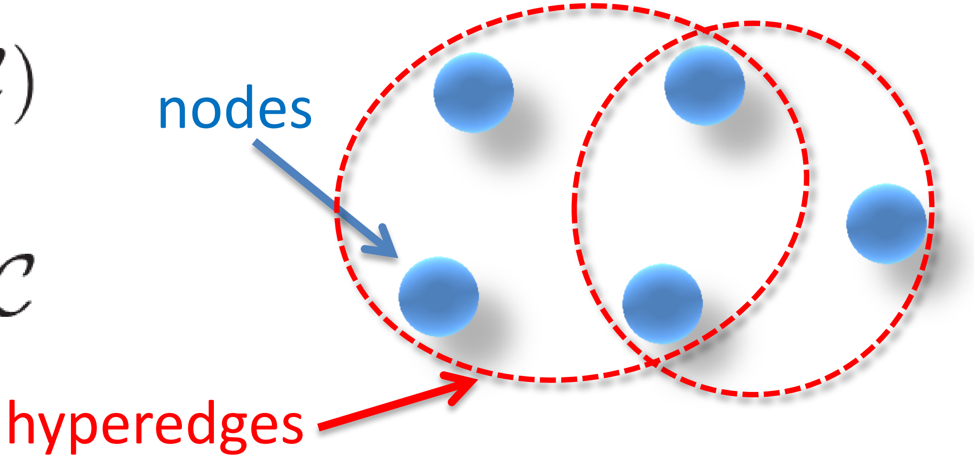
- Key task: inference/optimization for CRFs/MRFs
- Extensive research for more than 20 years
- Lots of progress
- Many state-of-the-art methods:
  - Graph-cut based algorithms
  - Message-passing methods
  - LP relaxations
  - Dual Decomposition
  - ….

MAP inference for CRFs/MRFs

- Hypergraph $G = (\mathcal{V}, \mathcal{C})$
  - Nodes $\mathcal{V}$
  - Hyperedges/cliques $\mathcal{C}$



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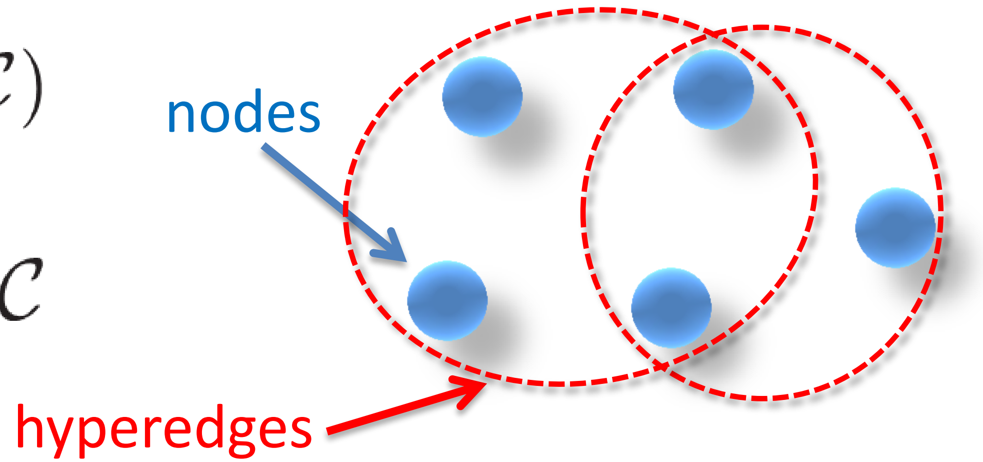

- High-order MRF energy minimization problem

$$\mathrm{MRF}_G(\mathbf{U}, \mathbf{H}) \equiv \min_{\mathbf{x}} \sum_{q \in \mathcal{V}} U_q(x_q) + \sum_{c \in \mathcal{C}} H_c(\mathbf{x}_c)$$

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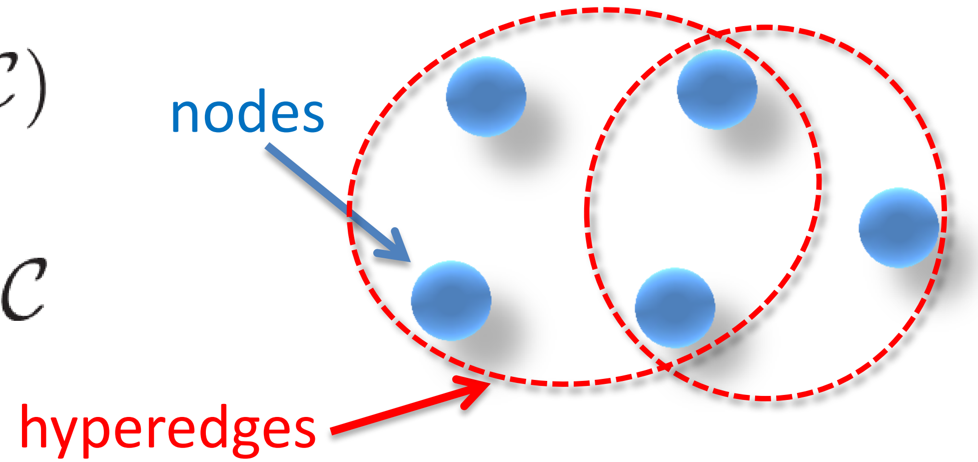

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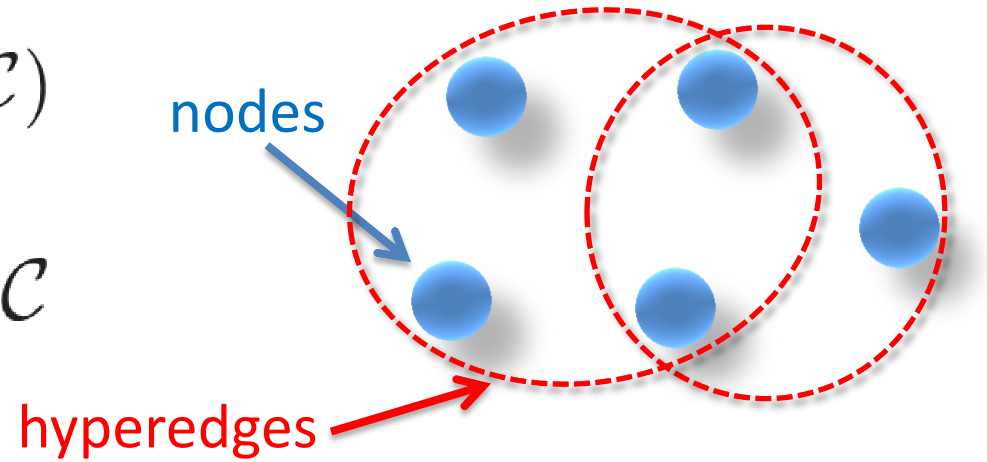

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CRF training

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- Through training
  - Parameterize potentials by **w**
  - Use training data to learn correct **w**

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- Characteristic example of structured output learning [Taskar], [Tsochantaridis, Joachims]

CRF training

- Equally, if not more, important than MAP inference
  - Better optimize correct energy
    (even approximately)
  - Than optimize wrong energy exactly

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- Equally, if not more, important than MAP inference
  - Better optimize correct energy (even approximately)
  - Than optimize wrong energy exactly
- Becomes even more important as we move towards:
  - complex models
  - high-order potentials
  - lots of parameters
  - lots of training data

CRF training

$$f : Z \longrightarrow X$$

CRF training

input
space

$$f : Z \longrightarrow X$$

can contain any
kind of data

CRF training

input space   output space

$$f : Z \rightarrow X$$

can contain any kind of data

CRF variables (structured object)

CRF training

input space

output space

$$f : Z \rightarrow X$$

can contain any kind of data

CRF variables (structured object)

Hereafter, we will use:

- symbol z to denote elements of space Z
- symbol x to denote elements of space X

CRF training

- Stereo matching:
  - Z: left, right image
  - X: disparity map

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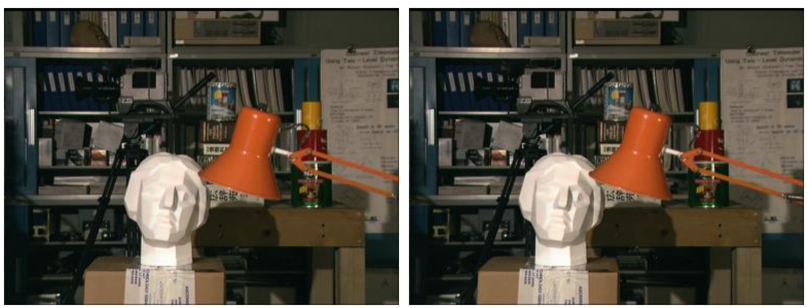

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X

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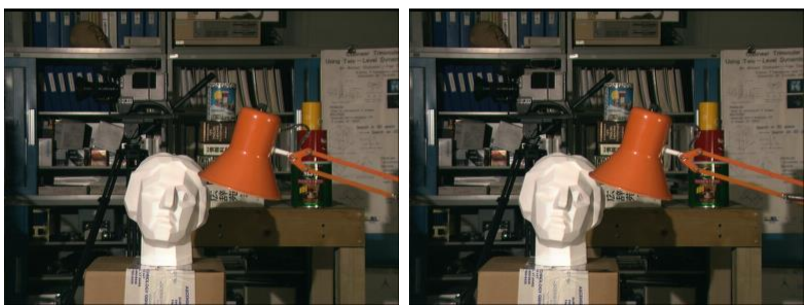

Z

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$$f = \arg\min_{\mathbf{x}} \mathrm{MRF}_G(\mathbf{x}; \mathbf{u}, \mathbf{h})$$

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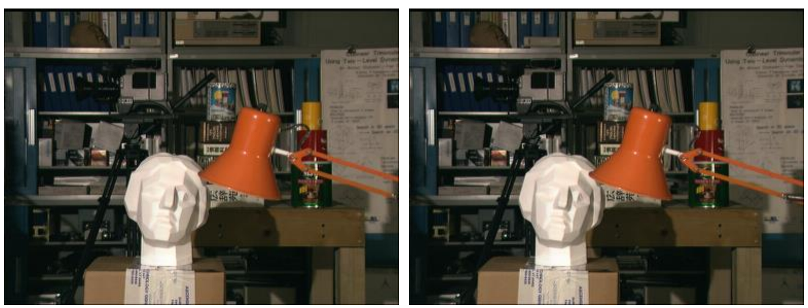

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Goal of training:
estimate proper **w**

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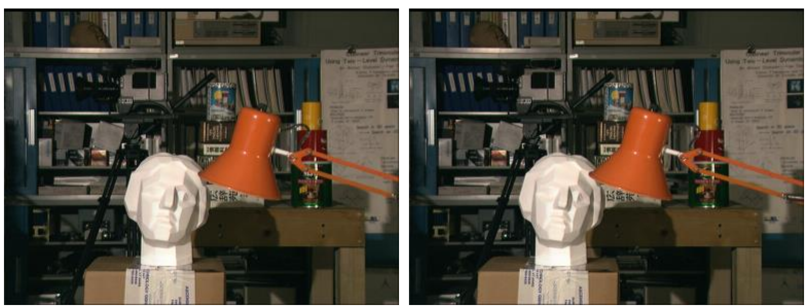

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$$f = \arg\min_{\mathbf{x}} \mathrm{MRF}_G(\mathbf{x}; \mathbf{u}, \mathbf{h})$$

parameterized by **w**

CRF training

- Denoising:
  - Z: noisy input image
  - X: denoised output image

Goal of training:
estimate proper **w**

$f:$

Z

X

$$f = \arg\min_{\mathbf{x}} \mathrm{MRF}_G(\mathbf{x}; \mathbf{u}, \mathbf{h})$$

parameterized by **w**

CRF training

- Object detection:
  - Z: input image
  - X: position of object parts

Goal of training:

estimate proper **w**

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$$u_p^k(x_p) = \mathbf{w}^T g_p(x_p,\mathbf{z}^k),\;\; h_c^k(\mathbf{x}_c) = \mathbf{w}^T g_c(\mathbf{x}_c,\mathbf{z}^k)$$

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vector valued feature functions

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Learning formulations

Risk minimization

K training samples $\left\{(\mathbf{x}^k, \mathbf{z}^k)\right\}_{k=1}^{K}$

Risk minimization

$$\min_{\mathbf{w}} \sum_{k=1}^{K} \Delta\left(\mathbf{x}^k, \hat{\mathbf{x}}^k\right)$$

K training samples $\left\{\left(\mathbf{x}^k, \mathbf{z}^k\right)\right\}_{k=1}^{K}$

Risk minimization

$$\hat{\mathbf{x}}^k = \arg\min_{\mathbf{x}} \mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k)$$

$$\min_{\mathbf{w}} \sum_{k=1}^{K} \Delta\left(\mathbf{x}^k, \hat{\mathbf{x}}^k\right)$$

K training samples $\left\{(\mathbf{x}^k, \mathbf{z}^k)\right\}_{k=1}^{K}$

Regularized Risk minimization

$$\hat{\mathbf{x}}^k = \arg\min_{\mathbf{x}} \mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k)$$

$$\min_{\mathbf{w}} R(\mathbf{w}) + \sum_{k=1}^{K} \Delta\left(\mathbf{x}^k, \hat{\mathbf{x}}^k\right)$$

$$R(\mathbf{w}) = ||\mathbf{w}||^2, \ \ ||\mathbf{w}||_1, \ \ \text{etc.}$$

Regularized Risk minimization

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Regularized Risk minimization

Replace Δ(.) with easier to handle upper bound L_G
(e.g., convex w.r.t. **w**)

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Regularized Risk minimization

$$\min_{\mathbf{w}} R(\mathbf{w}) + \sum_{k=1}^{K} L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right)$$

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Choice 1: Hinge loss

$$\min_{\mathbf{w}} R(\mathbf{w}) + \sum_{k=1}^{K} L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right)$$

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- Leads to max-margin learning

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Max-margin learning

$$\mathrm{MRF}_G(\mathbf{x}^k;\mathbf{w},\mathbf{z}^k) \leq \mathrm{MRF}_G(\mathbf{x};\mathbf{w},\mathbf{z}^k) - \Delta(\mathbf{x},\mathbf{x}^k)$$

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energy of ground truth	any other energy	desired margin

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$$\mathrm{MRF}_G(\mathbf{x}^k;\mathbf{w},\mathbf{z}^k) \leq \mathrm{MRF}_G(\mathbf{x};\mathbf{w},\mathbf{z}^k) - \Delta(\mathbf{x},\mathbf{x}^k) + \xi_k$$

energy of ground truth

any other energy

desired margin

slack

Max-margin learning

$$\min_{\mathbf{w}} \qquad \sum_k \xi_k$$

subject to the constraints:

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or equivalently

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Max-margin learning

CONSTRAINED

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Max-margin learning

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$\Updownarrow$ or equivalently

UNCONSTRAINED

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Choice 2: logistic loss

$$\min_{\mathbf{w}} R(\mathbf{w}) + \sum_{k=1}^{K} L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right)$$

$$\boxed{L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right) = \mathrm{MRF}_G(\mathbf{x}^k; \mathbf{w}, \mathbf{z}^k) + \underbrace{\log \sum_{\mathbf{x}} e^{-\mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k)}}_{\text{partition function}}}$$

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- Can be shown to lead to **maximum likelihood learning**

Max-margin vs Maximum-likelihood

$$L_G\left(\mathbf{x}^k,\mathbf{z}^k;\mathbf{w}\right) = \overbrace{\mathrm{MRF}_G(\mathbf{x}^k;\mathbf{w},\mathbf{z}^k) - \min_{\mathbf{x}}\left(\mathrm{MRF}_G(\mathbf{x};\mathbf{w},\mathbf{z}^k) - \Delta(\mathbf{x},\mathbf{x}^k)\right)}^{\text{max-margin}}$$

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soft-max

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Solving the learning formulations

Maximum-likelihood learning

Maximum-likelihood learning

$$\min_{\mathbf{w}} \frac{\mu}{2}||\mathbf{w}||^2 + \sum_{k=1}^{K} L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right)$$

$$L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right) = \mathrm{MRF}_G(\mathbf{x}^k; \mathbf{w}, \mathbf{z}^k) + \log \underbrace{\sum_{\mathbf{x}} e^{-\mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k)}}_{\text{partition function}}$$

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- Differentiable & convex

Maximum-likelihood learning

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- Differentiable & convex
- Global optimum via e.g. gradient descent

Maximum-likelihood learning

$$\min_{\mathbf{w}} \frac{\mu}{2}||\mathbf{w}||^2 + \sum_{k=1}^{K} L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right)$$

$$L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right) = \mathrm{MRF}_G(\mathbf{x}^k; \mathbf{w}, \mathbf{z}^k) + \log \sum_{\mathbf{x}} e^{-\mathrm{MRF}_G(\mathbf{x};\mathbf{w},\mathbf{z}^k)}$$

gradient $\longrightarrow \nabla_{\mathbf{w}} = \mathbf{w} + \sum_k \left(g(\mathbf{x}^k, \mathbf{z}^k) - \sum_{\mathbf{x}} p(\mathbf{x}|w, \mathbf{z}^k) g(\mathbf{x}, \mathbf{z}^k) \right)$

Recall that: $\mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k) = \mathbf{w}^T g(\mathbf{x}, \mathbf{z}^k)$

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- Requires MRF probabilistic inference

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- Requires MRF probabilistic inference
- **NP-hard** (exponentially many $\mathbf{x}$): approximation via loopy-BP ???

Max-margin learning (UNCONSTRAINED)

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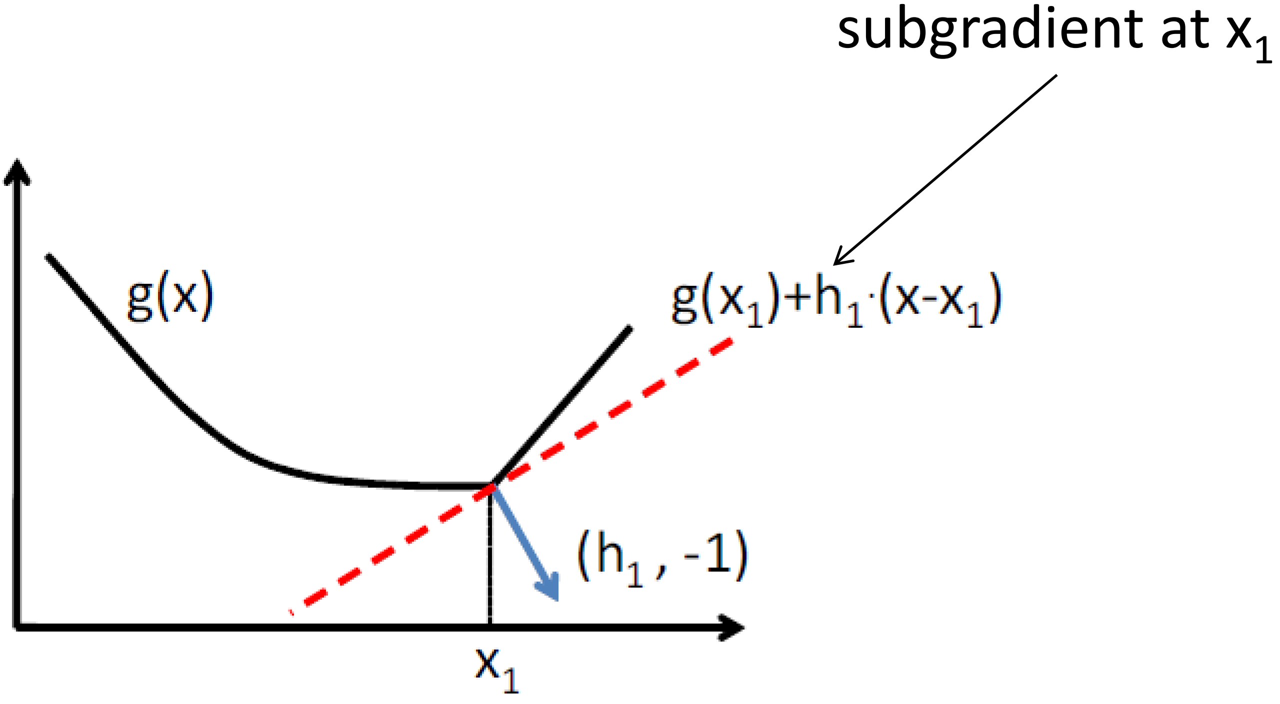
Subgradient



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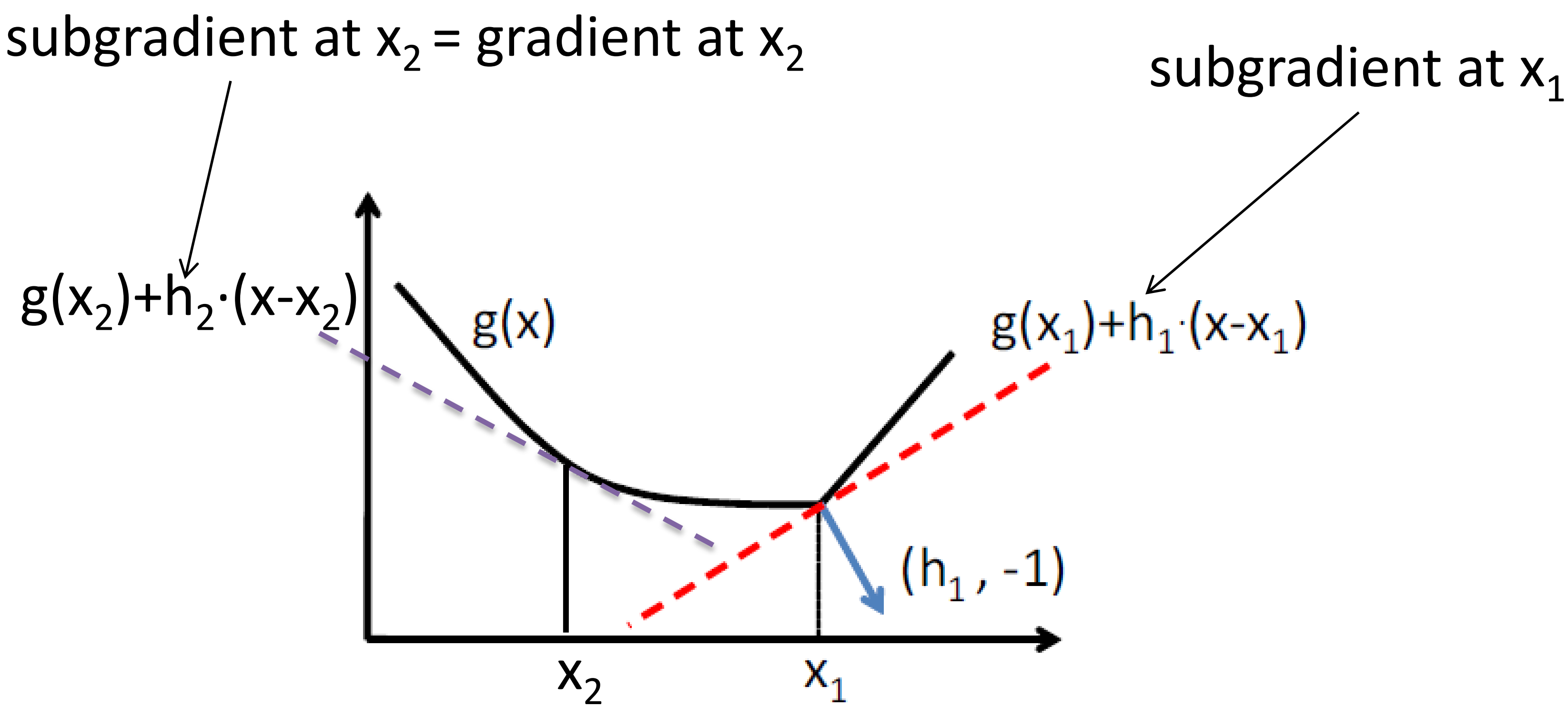

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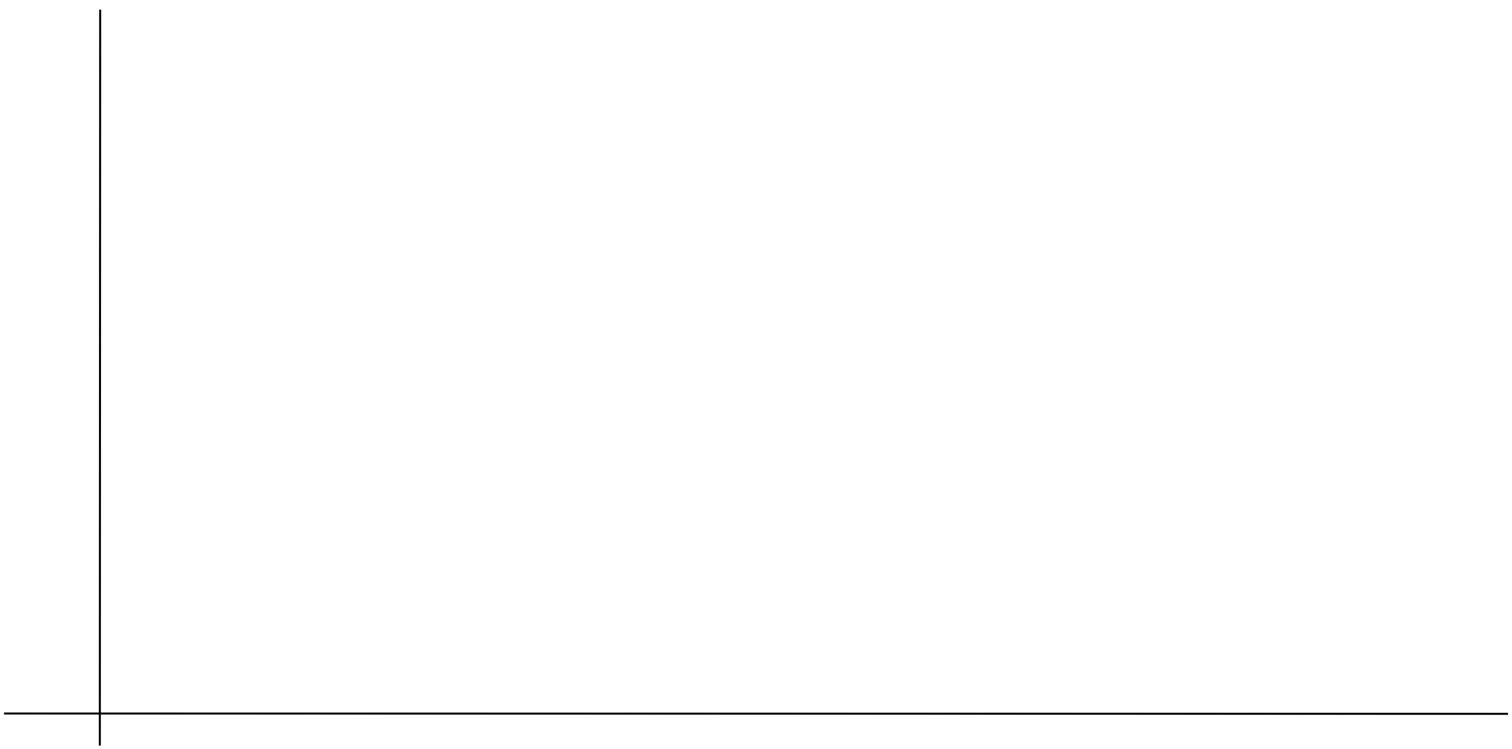

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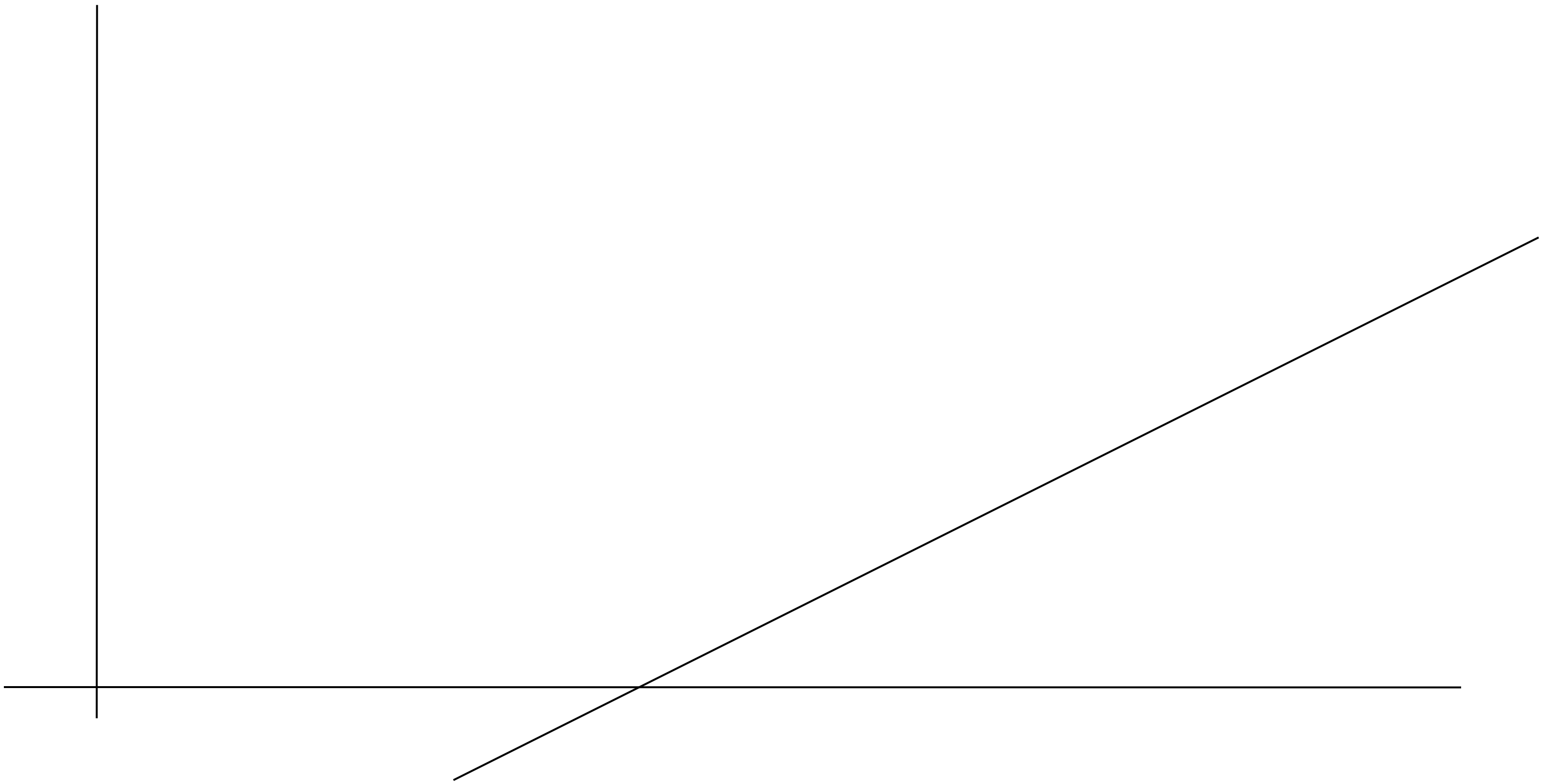

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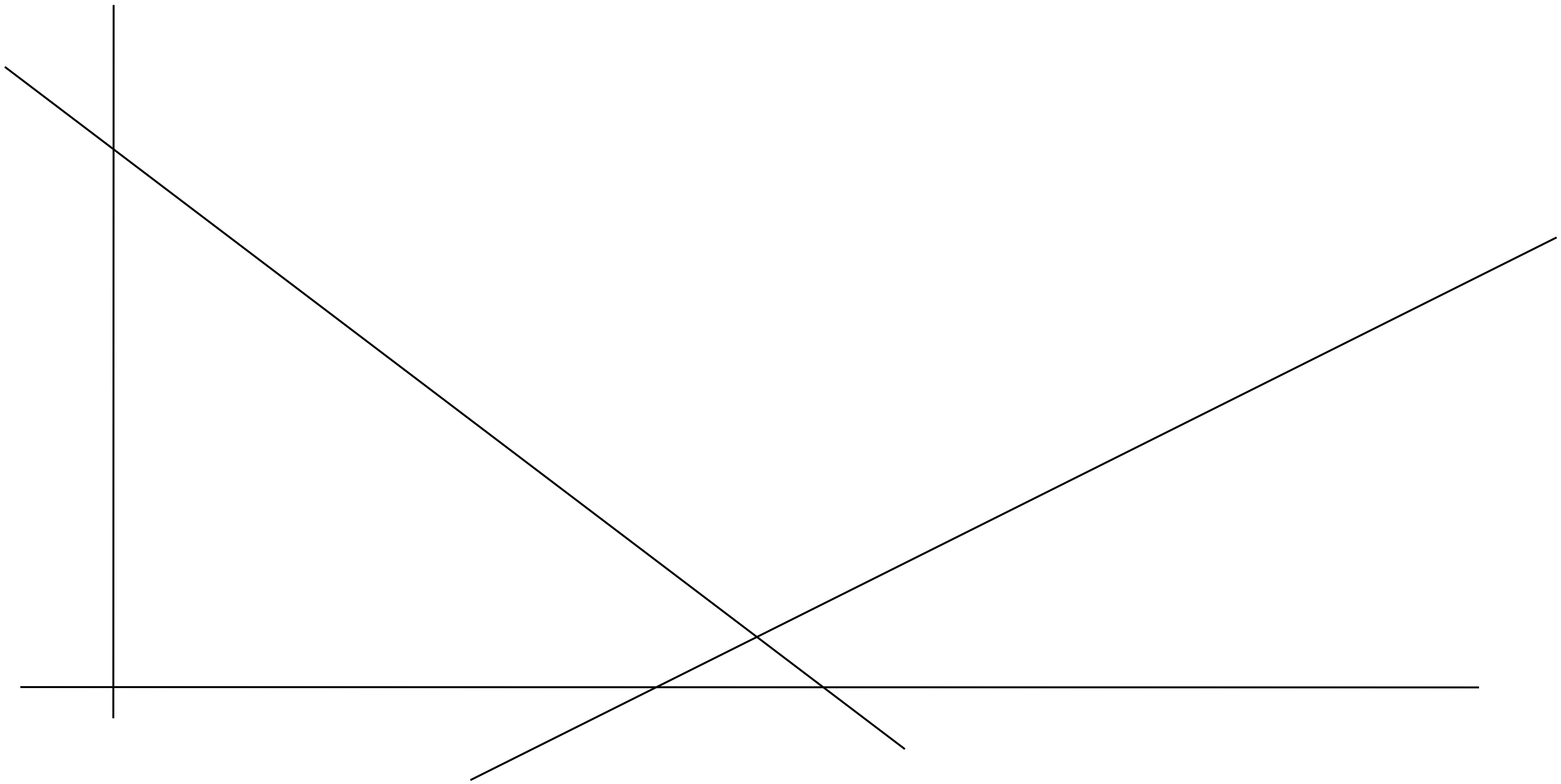

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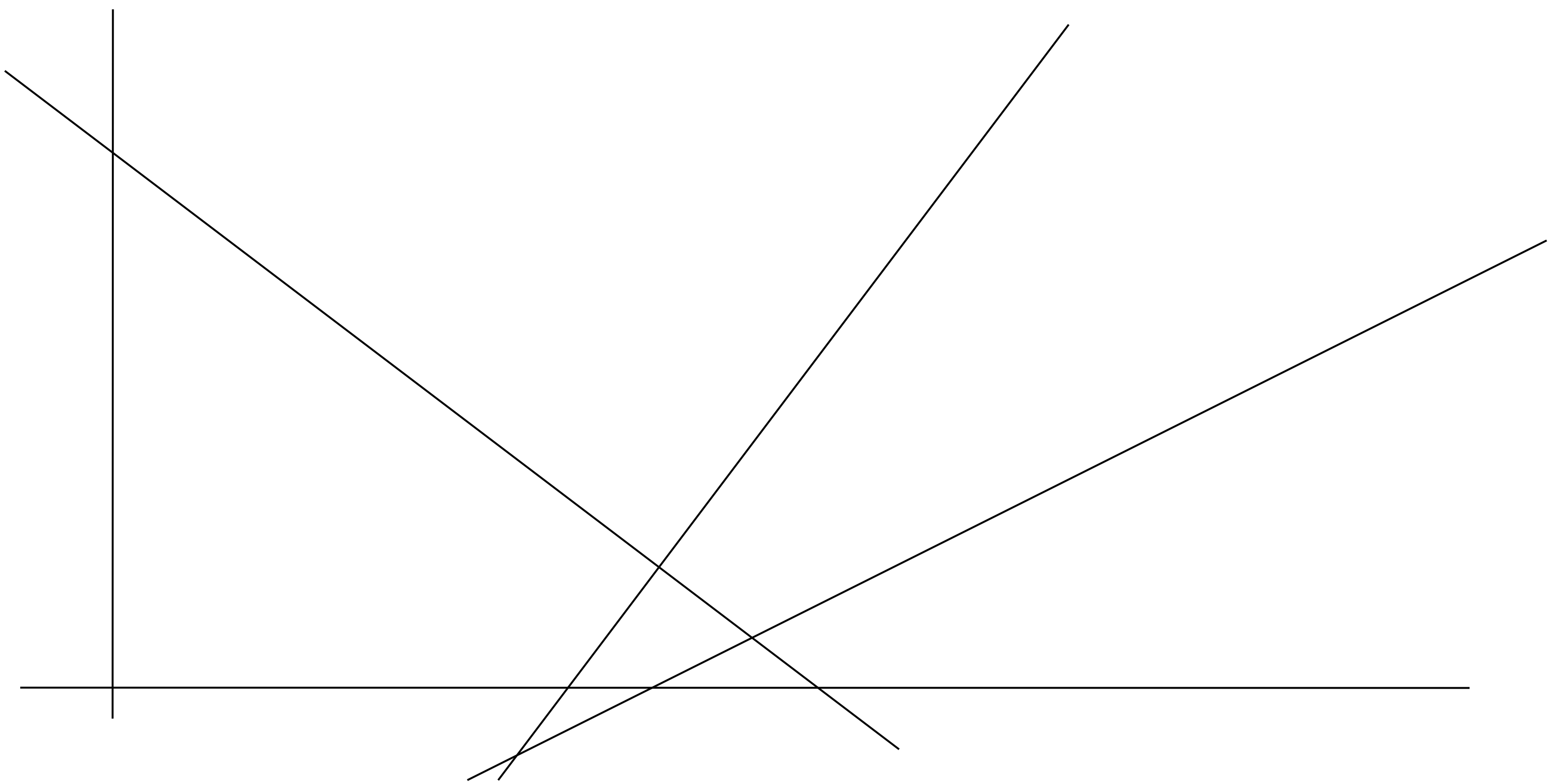

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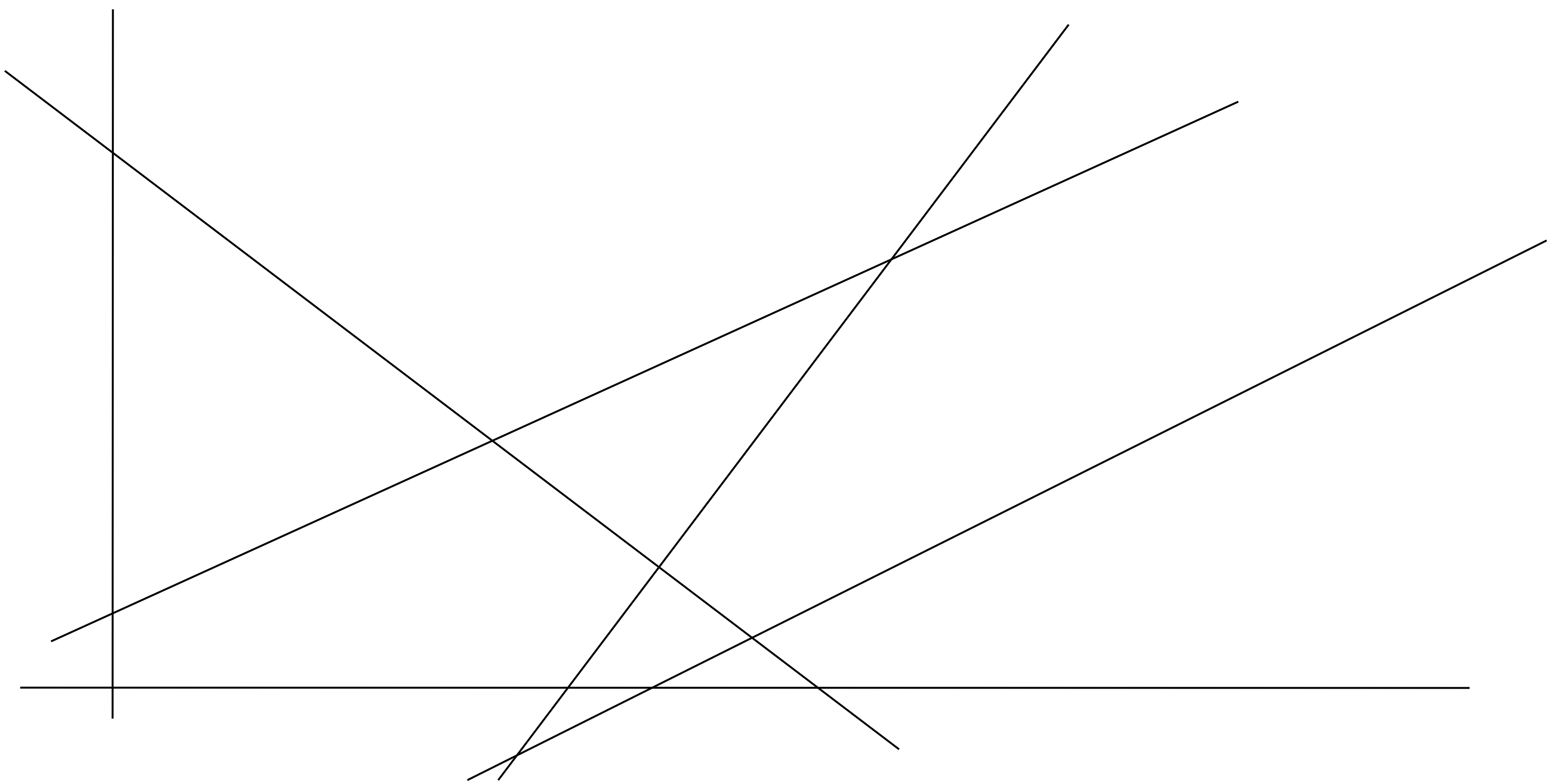

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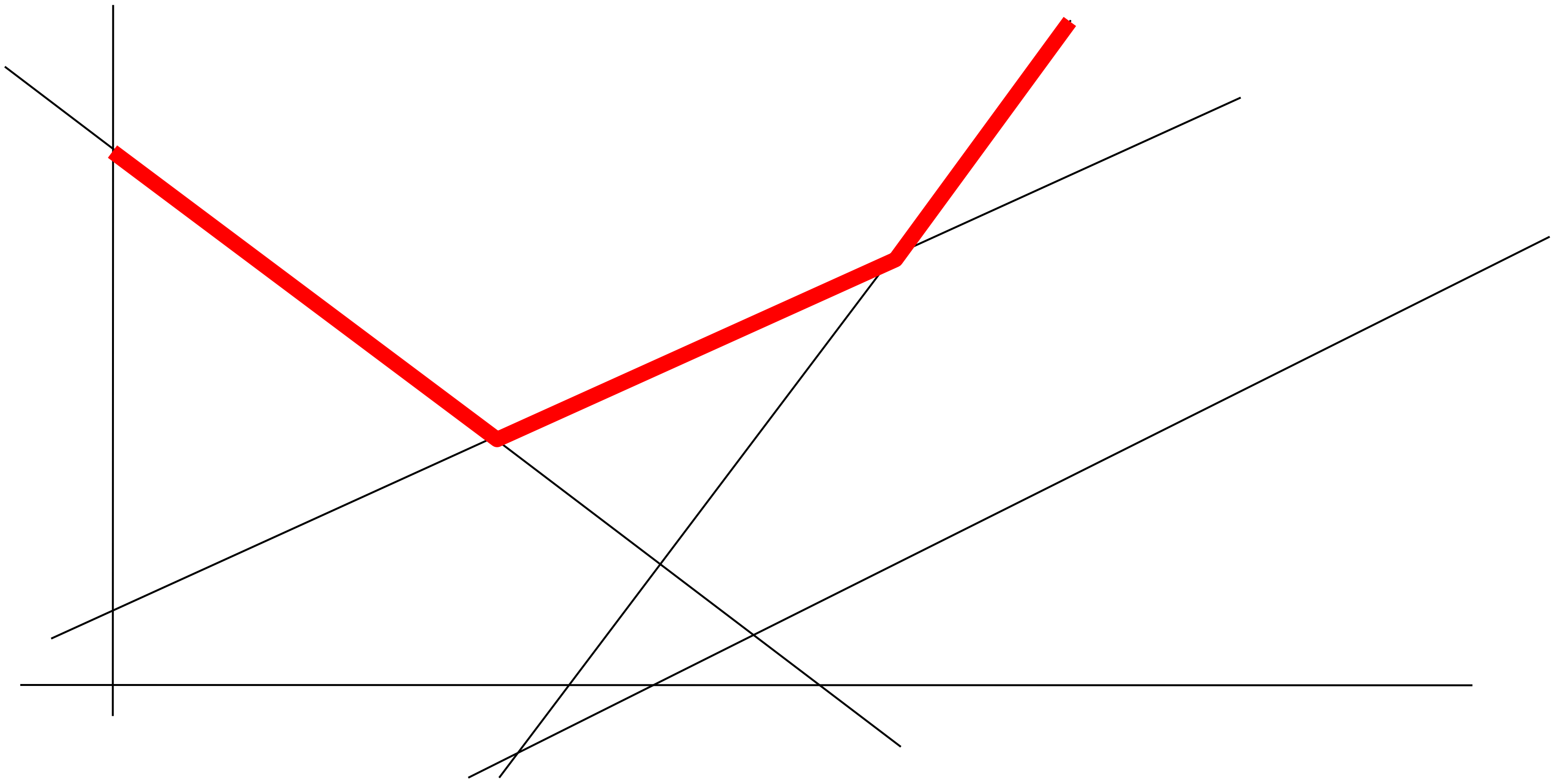

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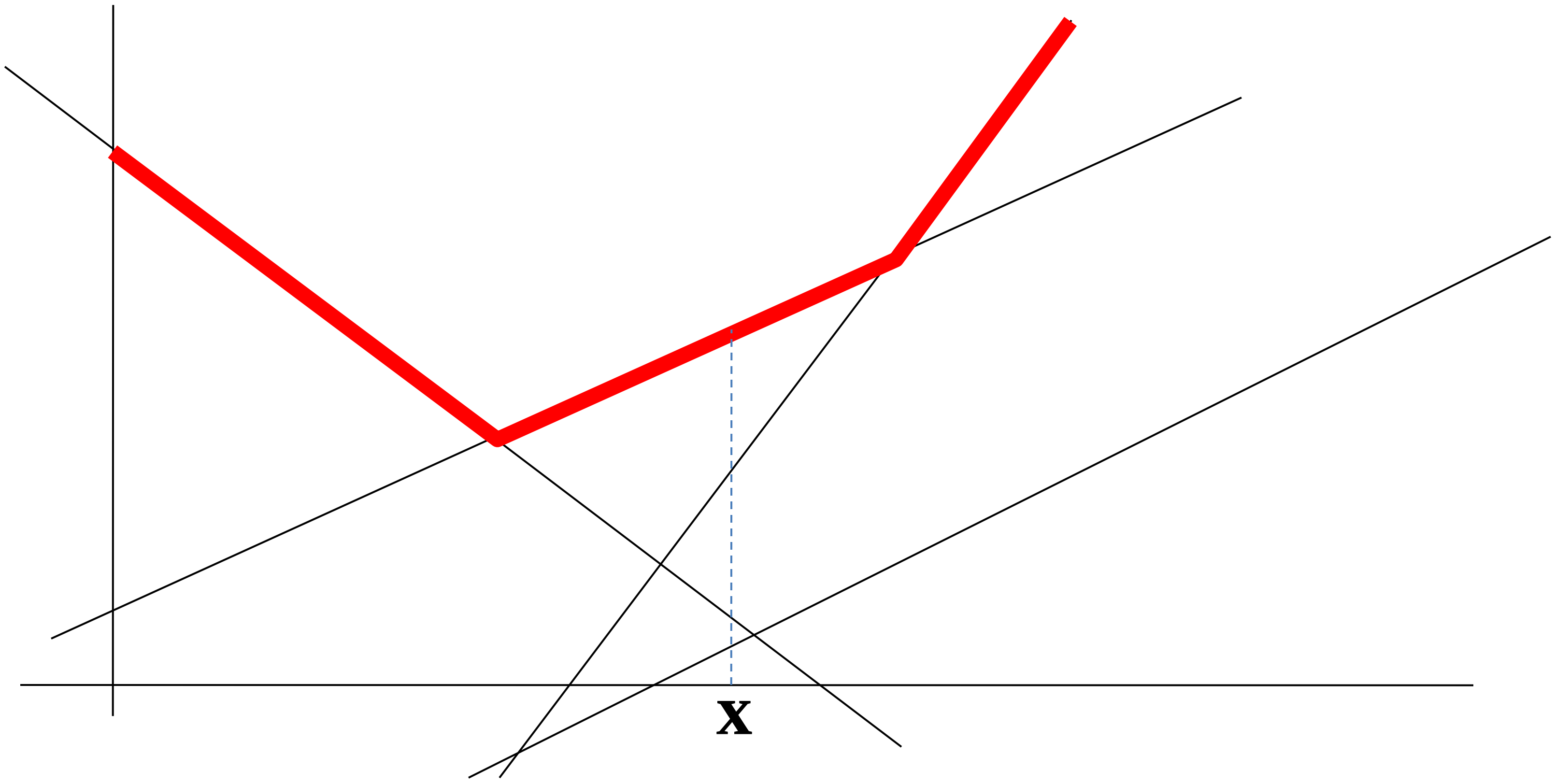

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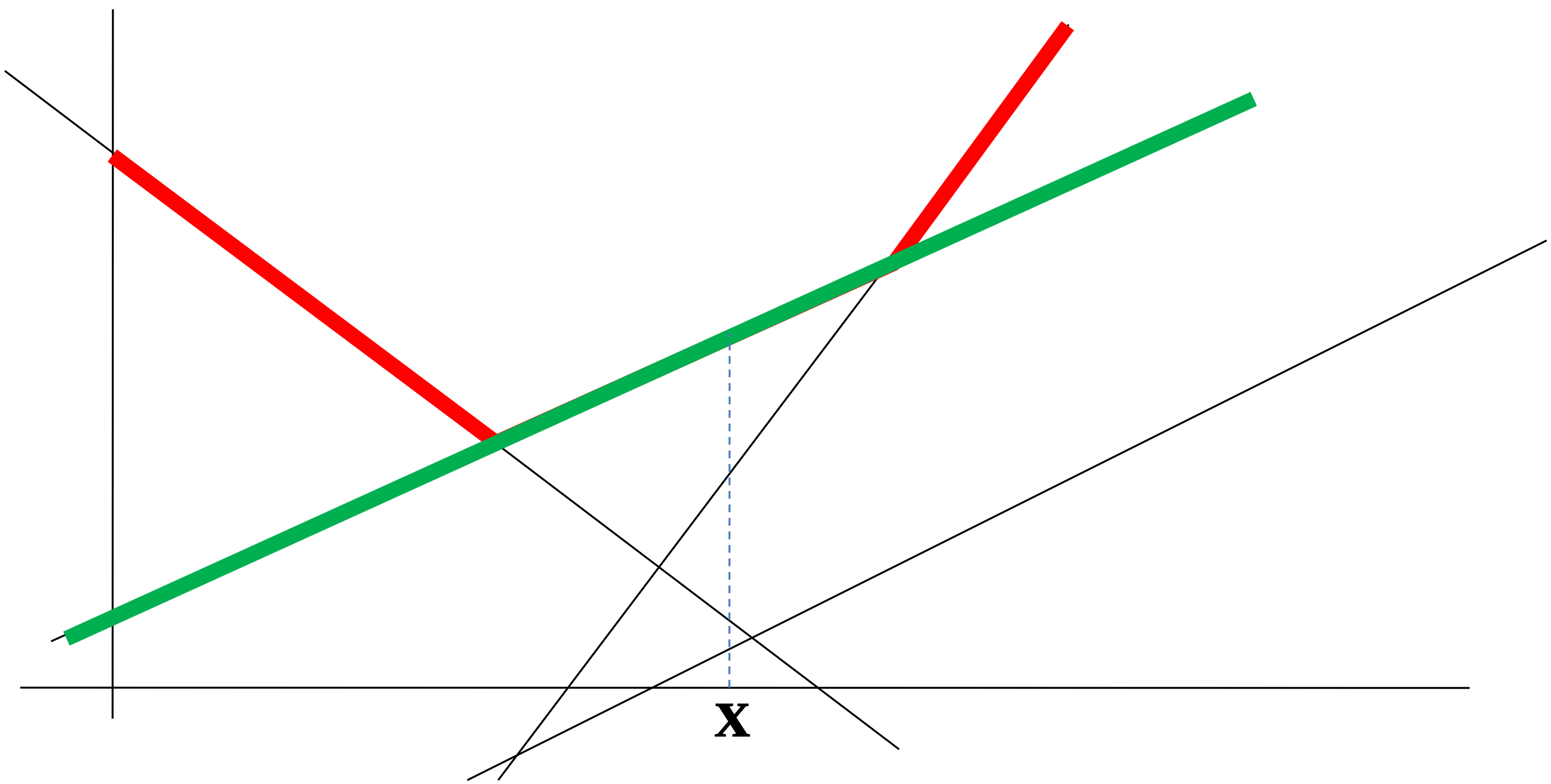

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Subgradient algorithm
Repeat 1. compute global minimizers $\hat{\mathbf{x}}^k$ at current $\mathbf{w}$ 2. compute **total subgradient** at current $\mathbf{w}$ 3. update $\mathbf{w}$ by taking a step in the negative total subgradient direction **until convergence**

total subgr. = $\mathrm{subgradient}_{\mathbf{w}}[R(\mathbf{w})] + \sum_k \left(g(\mathbf{x}^k, \mathbf{z}^k) - g(\hat{\mathbf{x}}^k, \mathbf{z}^k)\right)$

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Stochastic subgradient algorithm

Repeat

1. pick k at random
2. compute global minimizer $\hat{\mathbf{x}}^k$ at current $\mathbf{w}$
3. compute **partial subgradient** at current $\mathbf{w}$
4. update $\mathbf{w}$ by taking a step in the negative partial subgradient direction

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MRF-MAP estimation per iteration

partial subgradient = $\mathrm{subgradient}_{\mathbf{w}}[R(\mathbf{w})] + g(\mathbf{x}^k, \mathbf{z}^k) - g(\hat{\mathbf{x}}^k, \mathbf{z}^k)$

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(unfortunately NP-hard)

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- Quadratic program (great!)
- But exponentially many constraints (not so great)

Max-margin learning (CONSTRAINED)

- What if we use only a small number of constrains?
  - Resulting QP can be solved
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  - only few constraints **active** at optimal solution !! (variables much fewer than constraints)
  - Given the active constraints, rest can be ignored
  - Then let's try to find them!

Constraint generation

1. Start with some constraints

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3. Check if solution is feasible w.r.t. to **all** constraints
4. If yes, we are done!
5. If no, pick a violated constraint and add it to the current set of constraints. Go to step 2

   (optionally, we can also remove inactive constraints)

Constraint generation

- **Key issue:** we must always be able to find a violated constraint if one exists

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- Recall the constraints for max-margin learning

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- To find violated constraint, we therefore need to compute:

$$\hat{\mathbf{x}}^k = \arg\min_{\mathbf{x}} \left(\mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k) - \Delta(\mathbf{x}, \mathbf{x}^k) \right)$$

(just like subgradient method!)

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1. Initialize set of constraints C to empty

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5. If no new constraint was added then terminate. Otherwise go to step 2.

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MRF-MAP estimation **per sample**
(unfortunately **NP-hard**)

Max-margin learning (CONSTRAINED)

$$\min_{\mathbf{w}} \frac{\mu}{2}||\mathbf{w}||^2 + \sum_k \xi_k$$

subject to the constraints:

$$\mathrm{MRF}_G(\mathbf{x}^k; \mathbf{w}, \mathbf{z}^k) \leq \mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k) - \Delta(\mathbf{x}, \mathbf{x}^k) + \xi_k$$

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- Alternatively, we can solve above QP in the **dual domain**
- dual variables $\leftrightarrow$ primal constraints
- Too many variables, but most of them zero at optimal solution
- Use a **working-set** method
(essentially dual to constraint generation)

CRF Training via Dual Decomposition [CVPR 2011]

CRF training

- Existing max-margin (maximum likelihood) methods:

CRF training

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  - computational efficiency ?
  - accuracy ?
  - theoretical guarantees/properties ?
- **Key issue**: can we more properly exploit CRF structure during training?

CRF Training via Dual Decomposition

- Efficient max-margin training method

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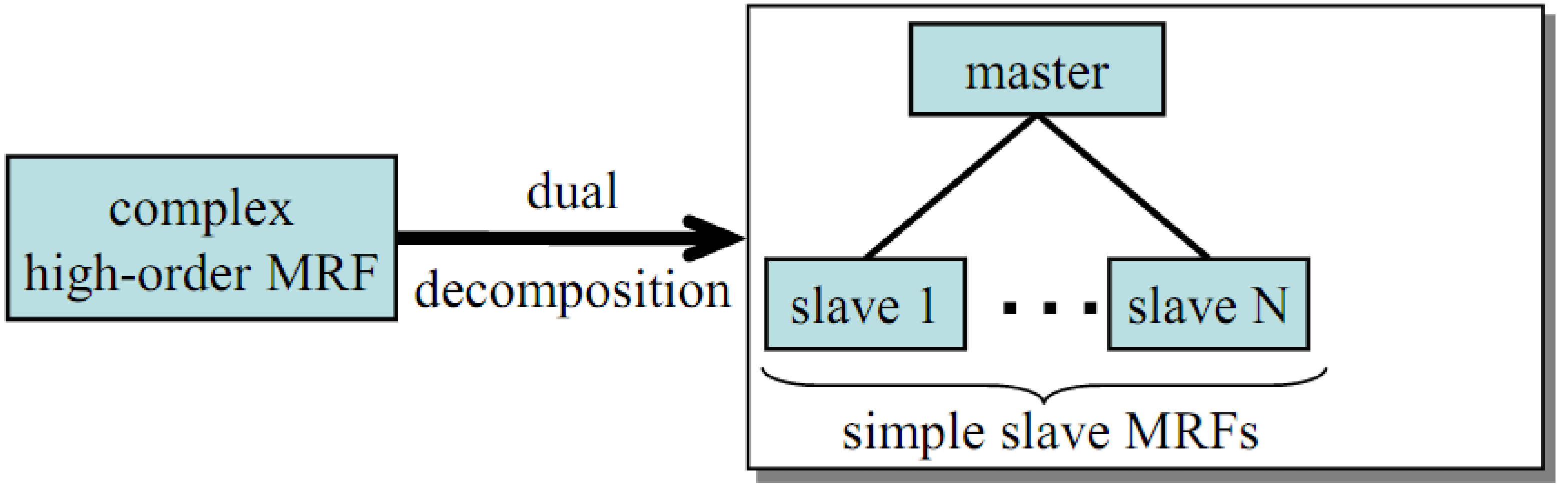
- Handles arbitrary **pairwise or higher-order** CRFs
- Uses **very efficient** projected subgradient learning scheme
- Allows hierarchy of structured prediction learning algorithms of **increasing accuracy**
- Very **flexible and adaptable**
    - Easily adjusted to fully exploit additional structure in any class of CRFs (no matter if they contain very high order cliques or not)

Dual Decomposition for MRF Optimization
(short review)

MRF Optimization via Dual Decomposition

- Very general framework for MAP inference [Komodakis et al. ICCV07, PAMI11]



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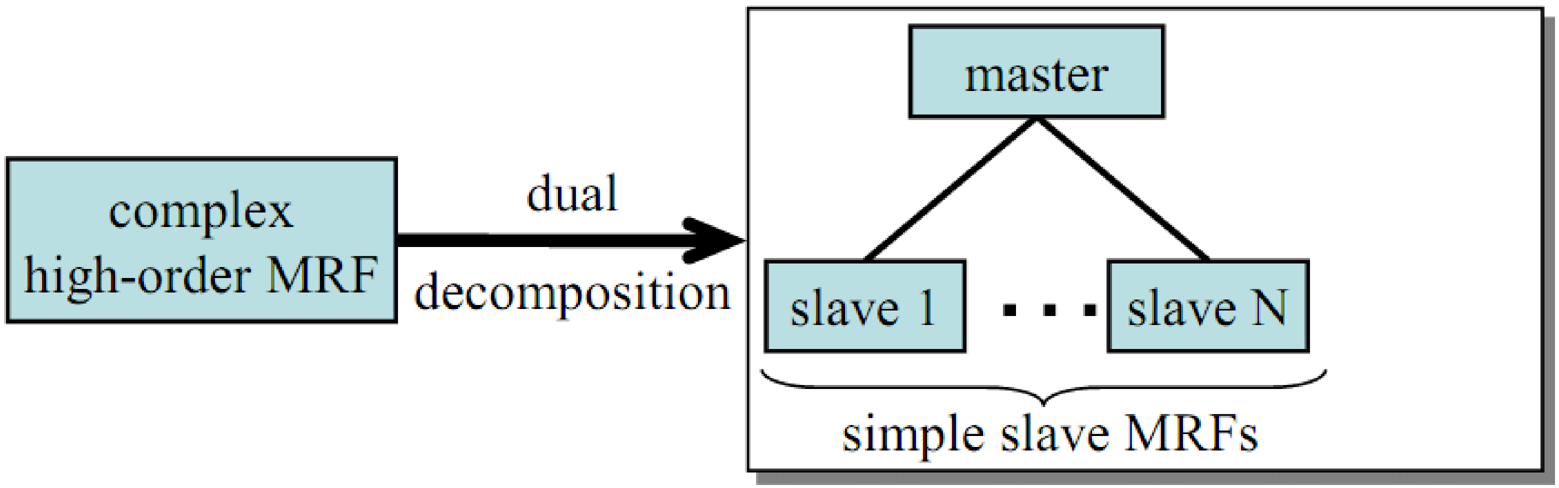

- Master = coordinator (has global view)
  Slaves = subproblems (have only local view)

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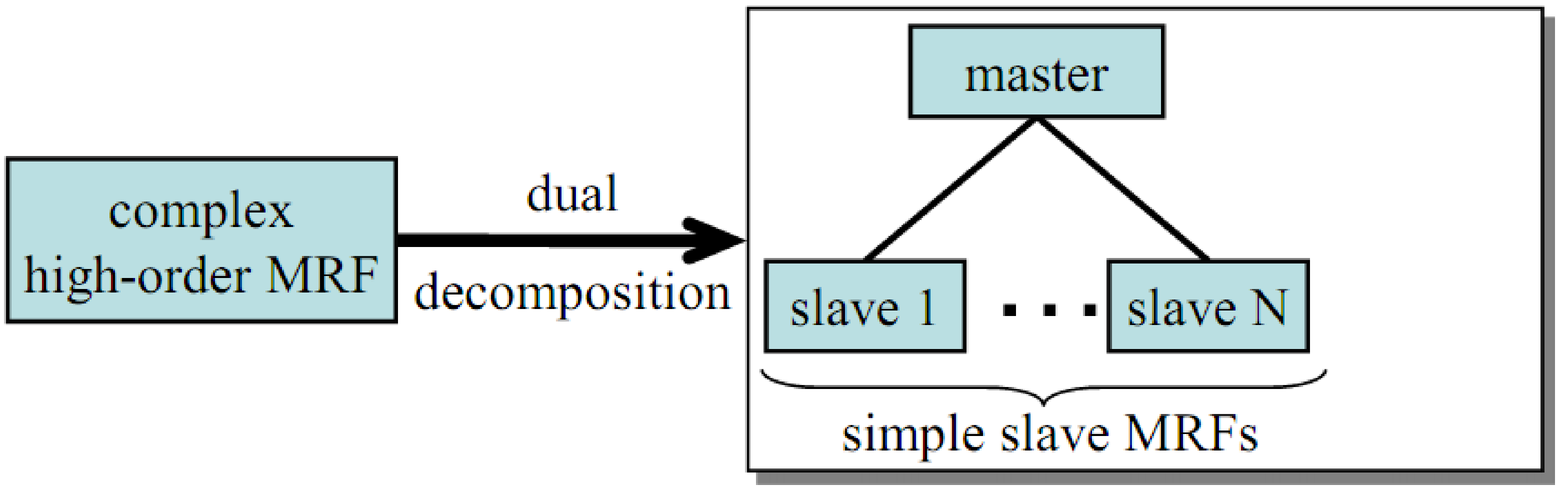

- Master = $\mathrm{MRF}_G(\mathbf{u}, \mathbf{h}) \longleftarrow$ (MAP-MRF on hypergraph *G*)

  = min $\mathrm{MRF}_G(\mathbf{x}; \mathbf{u}, \mathbf{h}) := \sum_{p \in \mathcal{V}} u_p(x_p) + \sum_{c \in \mathcal{C}} h_c(\mathbf{x}_c)$

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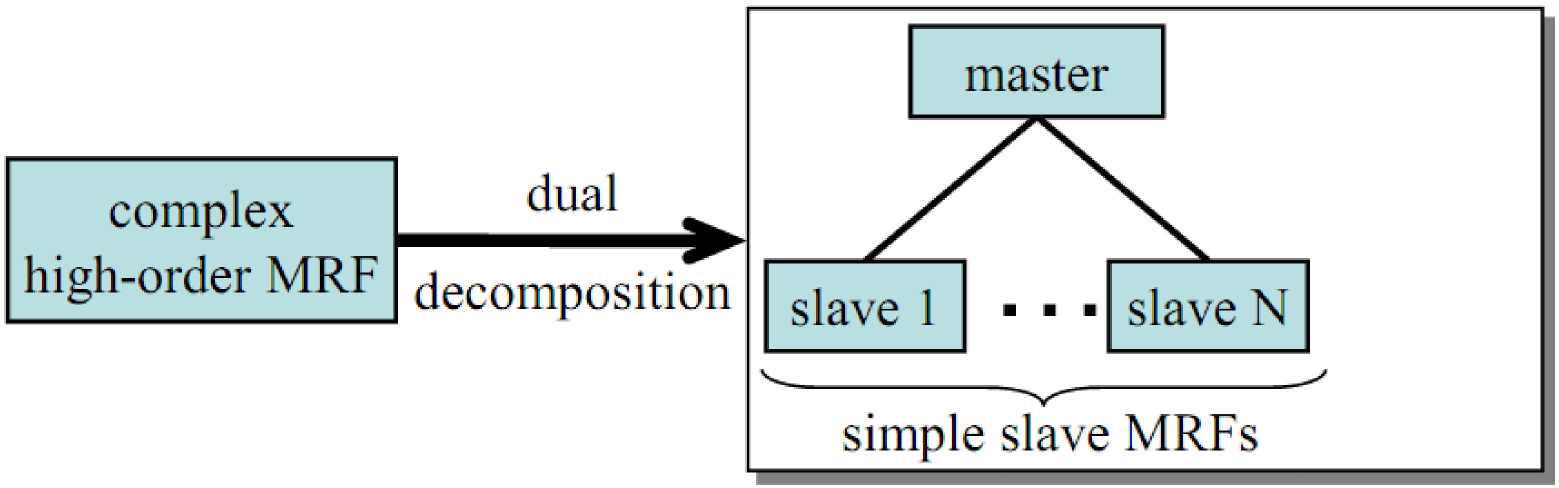

- Set of slaves = $\{\mathrm{MRF}_{G_i}(\boldsymbol{\theta}^i, \mathbf{h})\}$
(MRFs on sub-hypergraphs G_i whose union covers G)
- Many other choices possible as well

MRF Optimization via Dual Decomposition

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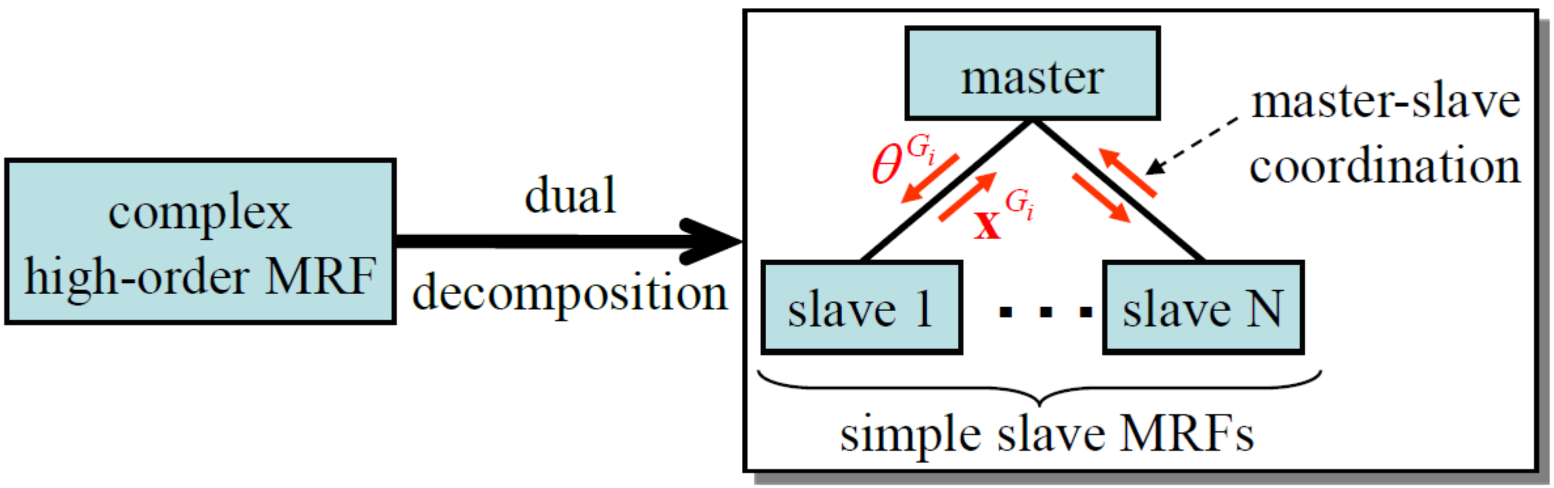

- Optimization proceeds in an iterative fashion via **master-slave coordination**

MRF Optimization via Dual Decomposition

Set of slave MRFs

$$\{\mathrm{MRF}_{G_i}(\boldsymbol{\theta}^i, \mathbf{h})\}$$

convex dual relaxation

$$\mathrm{DUAL}_{\{G_i\}}(\mathbf{u}, \mathbf{h}) = \max_{\{\boldsymbol{\theta}^i\}} \sum_i \mathrm{MRF}_{G_i}(\boldsymbol{\theta}^i, \mathbf{h})$$

$$\text{s.t.} \sum_{i \in \mathcal{I}_p} \theta_p^i(\cdot) = u_p(\cdot)$$

For each choice of slaves, master solves (possibly different) dual relaxation

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- Sum of slave energies = lower bound on MRF optimum

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For each choice of slaves, master solves (possibly different) dual relaxation

- Sum of slave energies = lower bound on MRF optimum
- Dual relaxation = maximum such bound

MRF Optimization via Dual Decomposition

Set of slave MRFs

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Choosing more difficult slaves $\Longrightarrow$ tighter lower bounds

$\Longrightarrow$ tighter dual relaxations

Dual Decomposition for MRF Optimization
(short review finished)

Max-margin learning via dual decomposition

$$\min_{\mathbf{w}} R(\mathbf{w}) + \sum_{k=1}^{K} L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right)$$

$$L_G\left(\mathbf{x}^k, \mathbf{z}^k; \mathbf{w}\right) = \mathrm{MRF}_G(\mathbf{x}^k; \mathbf{w}, \mathbf{z}^k) - \min_{\mathbf{x}} \left(\mathrm{MRF}_G(\mathbf{x}; \mathbf{w}, \mathbf{z}^k) - \Delta(\mathbf{x}, \mathbf{x}^k)\right)$$

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loss-augmented potentials

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$$\Downarrow$$

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Problem

Learning objective intractable due to this term

Max-margin learning via dual decomposition

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Solution: approximate this term with dual relaxation from decomposition $\{G_i = (\mathcal{V}_i, \mathcal{C}_i)\}_{i=1}^{N}$

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Max-margin learning via dual decomposition

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$$\text{s.t.} \quad \sum_{i \in \mathcal{I}_p} \theta_p^{(i,k)}(\cdot) = \bar{u}_p^k(\cdot)$$

Max-margin learning via dual decomposition

now

$$\min_{\mathbf{w},\{\boldsymbol{\theta}^{(i,k)}\}} R(\mathbf{w}) + \sum_k \sum_i L_{G_i}(\mathbf{x}^k, \boldsymbol{\theta}^{(i,k)}, \bar{\mathbf{h}}^k; \mathbf{w})$$

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before

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Max-margin learning via dual decomposition

now

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Essentially, training of complex CRF decomposed to parallel training of easy-to-handle slave CRFs !!!

Max-margin learning via dual decomposition

- Global optimum via projected subgradient method
  (slight variation of subgradient method)

Max-margin learning via dual decomposition

- Global optimum via projected subgradient method (slight variation of subgradient method)

Projected subgradient
Repeat 1. compute subgradient at current $\mathbf{w}$ 2. update $\mathbf{w}$ by taking a step in the negative subgradient direction 3. project into feasible set **until convergence**

Projected subgradient learning algorithm

- **Input:**
  - K training samples $\left\{(\mathbf{x}^k, \mathbf{z}^k)\right\}_{k=1}^{K}$
  - Hypergraph $G = (\mathcal{V}, \mathcal{C})$
    (in general hypergraphs can vary per sample)
  - Vector valued feature functions $\{g_p(\cdot,\cdot)\}, \{g_c(\cdot,\cdot)\}$

Projected subgradient learning algorithm

$\forall k,$ choose decomposition $\{G_i = (\mathcal{V}_i, \mathcal{C}_i)\}_{i=1}^{N}$ of hypergraph G

$\forall k, i,$ initialize $\boldsymbol{\theta}^{(i,k)}$ so as to satisfy $\sum_{i \in \mathcal{I}_p} \theta_p^{(i,k)}(\cdot) = \bar{u}_p^k(\cdot)$

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until convergence

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repeat

// ***optimize slave MRFs***

$\forall k, i,$ compute minimizer $\hat{\mathbf{x}}^{(i,k)} = \arg\min_{\mathbf{x}} \mathrm{MRF}_{G_i}(\mathbf{x}; \boldsymbol{\theta}^{(i,k)}, \bar{\mathbf{h}}^k)$

until convergence

Projected subgradient learning algorithm

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// optimize slave MRFs

$\forall k, i,$ compute minimizer $\hat{\mathbf{x}}^{(i,k)} = \arg\min_{\mathbf{x}} \mathrm{MRF}_{G_i}(\mathbf{x}; \boldsymbol{\theta}^{(i,k)}, \bar{\mathbf{h}}^k)$

// update $\mathbf{w}$

$\mathbf{w} \leftarrow \mathbf{w} - \alpha_t \cdot d\mathbf{w}$ fully specified from $\hat{\mathbf{x}}^{(i,k)}$

until convergence

Projected subgradient learning algorithm

$\forall k,$ choose decomposition $\{G_i = (\mathcal{V}_i, \mathcal{C}_i)\}_{i=1}^{N}$ of hypergraph G

$\forall k, i,$ initialize $\boldsymbol{\theta}^{(i,k)}$ so as to satisfy $\sum_{i \in \mathcal{I}_p} \theta_p^{(i,k)}(\cdot) = \bar{u}_p^k(\cdot)$

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(we only need to know how to optimize slave MRFs !!)

Projected subgradient learning algorithm

- **Incremental subgradient** version:
  - Same as before but considers subset of slaves per iteration
  - Subset chosen
    - deterministically or
    - randomly (**stochastic subgradient**)
  - Further improves computational efficiency
  - Same optimality guarantees & theoretical properties

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pick k

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  - ✓ Slave problems freely chosen by the user
  - ✓ Easily adaptable to further exploit special structure of any class of CRFs

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- $\{G_i\} < \{\tilde{G}_j\}$

  (hierarchy of learning algorithms)

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- $G_{\text{single}} = \{G_c\}_{c \in \mathcal{C}}$ denotes following decomposition:
  - One slave per clique $c \in \mathcal{C}$
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- Corresponding dual relaxation is an LP
  - Generalizes well known LP relaxation for pairwise MRFs (at the core of most state-of-the-art methods)

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- We are essentially adapting decomposition to exploit the structure of the problem at hand

Choice of decompositions $\{G_i\}$

- But we can do better if CRFs have special structure…

- E.g., **pattern-based** high-order potentials (for a clique c) [Komodakis & Paragios CVPR09]

$$H_c(\mathbf{x}) = \begin{cases} \psi_c(\mathbf{x}) & \text{if } \mathbf{x} \in \mathcal{P} \\ \psi_c^{\max} & \text{otherwise} \end{cases}$$

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- We only assume:
  - Set $\mathcal{P}$ is sparse
  - It holds $\psi_c(\mathbf{x}) \leq \psi_c^{\max},\ \forall \mathbf{x} \in \mathcal{P}$
  - No other restriction

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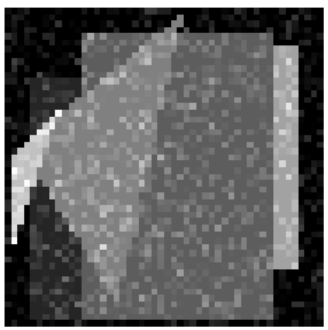
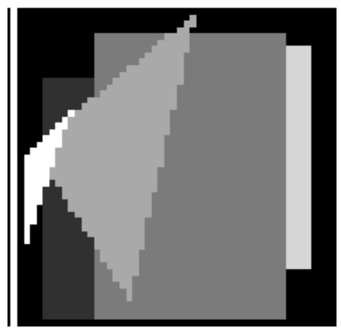
- But improvement in speed
  ($\text{DUAL}_{G_{\text{tree}}}$ converges faster than $\text{DUAL}_{G_{\text{single}}}$)

Image denoising

- Piecewise constant images



Z



X

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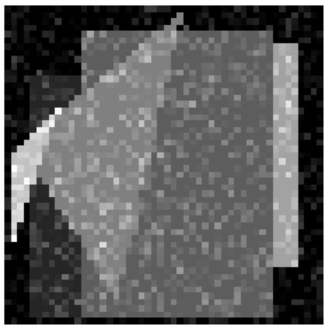

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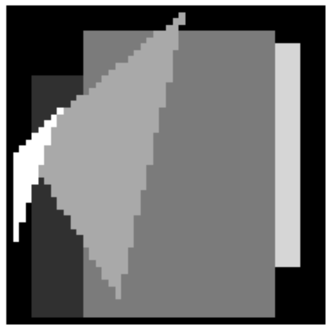

X

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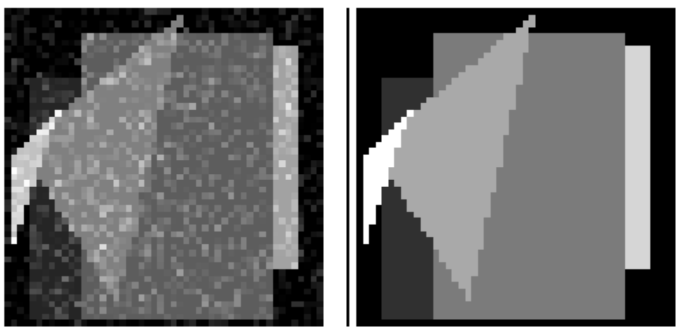

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- Goal: learn pairwise potential $V(\cdot)$

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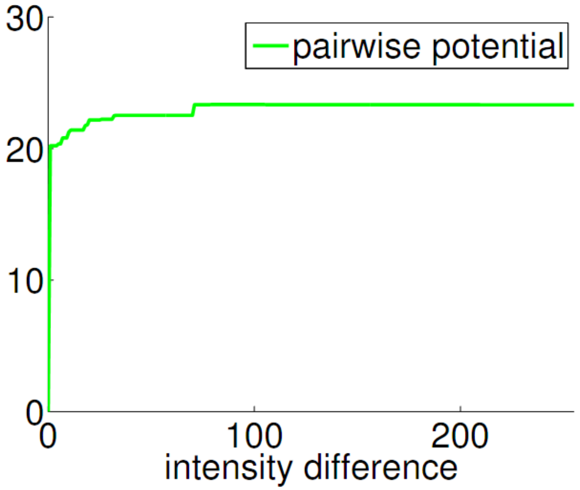

learnt potential

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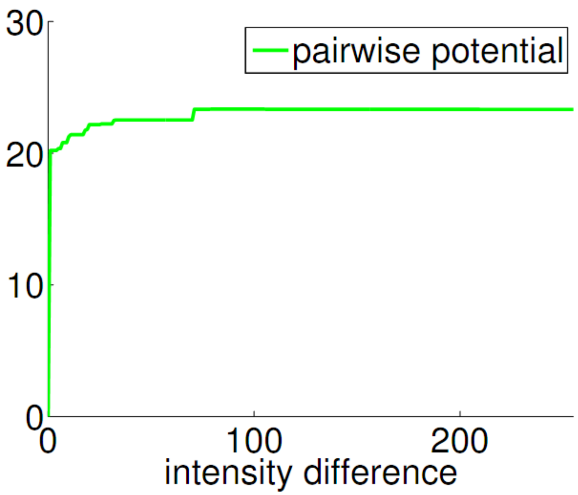

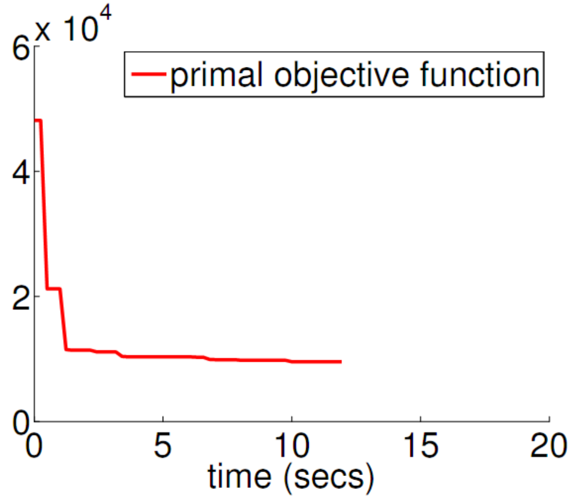

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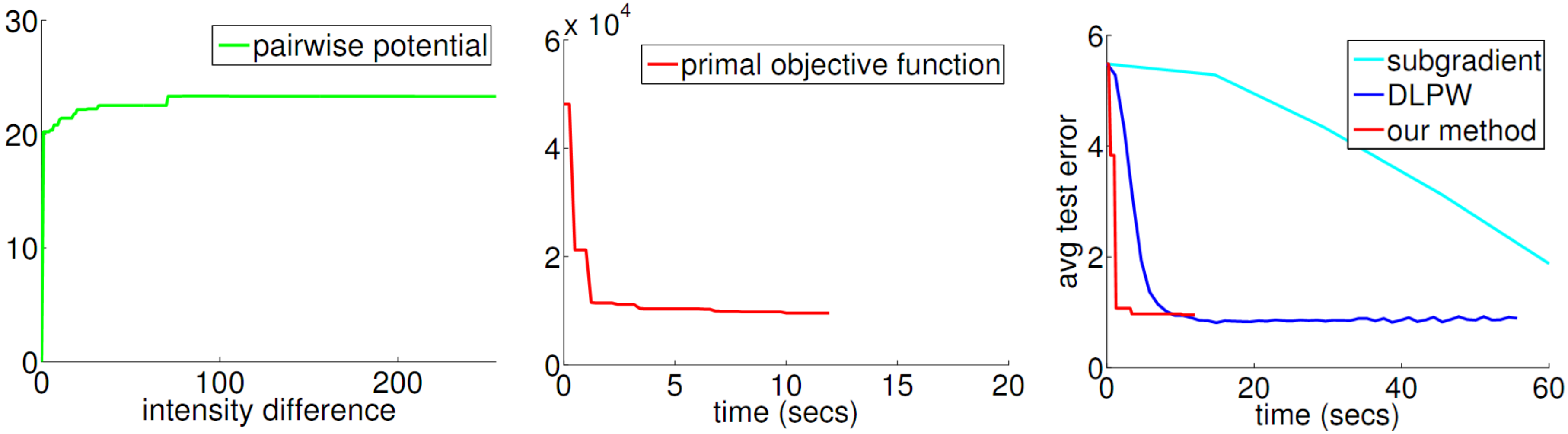

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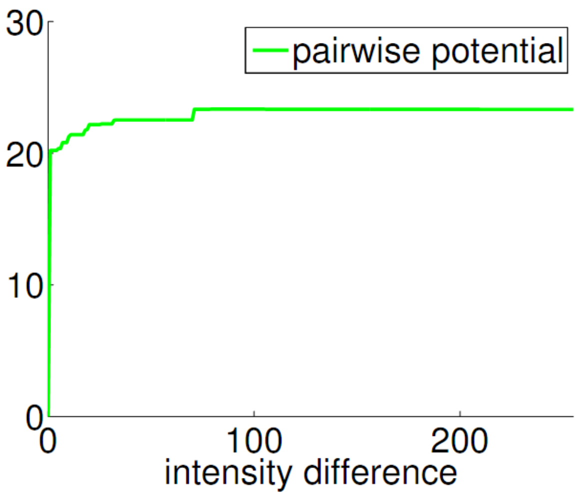

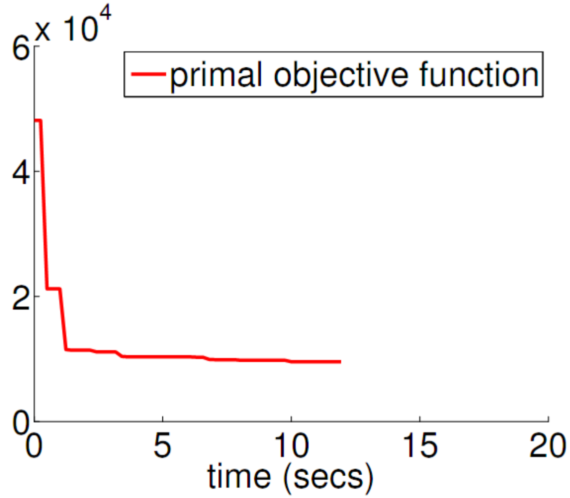

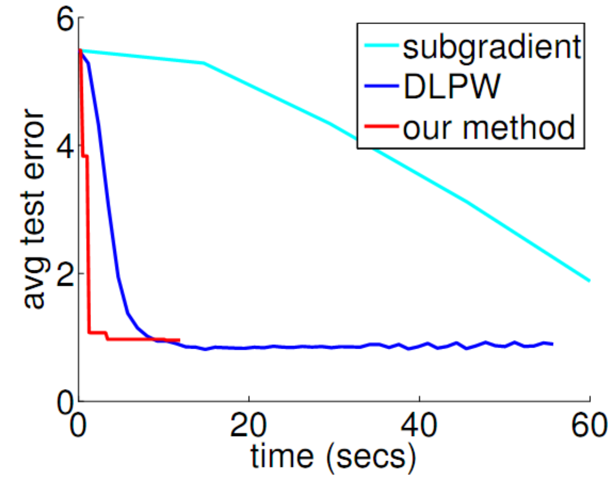

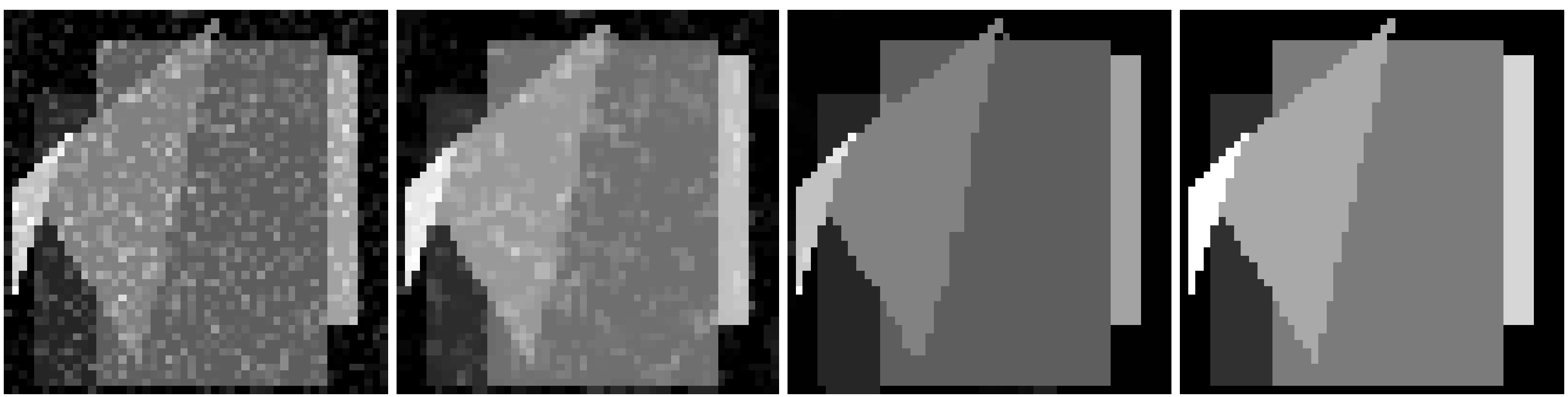

Stereo matching

- Potentials: $u_p^k\left(x_p\right) = \left| I^{left}\left(p\right) - I^{right}\left(p - x_p\right)\right|$

$$h_{pq}^k\left(x_p, x_q\right) = f\left(\left|\nabla I^{left}(p)\right|\right)\left[x_p \neq x_q\right]$$

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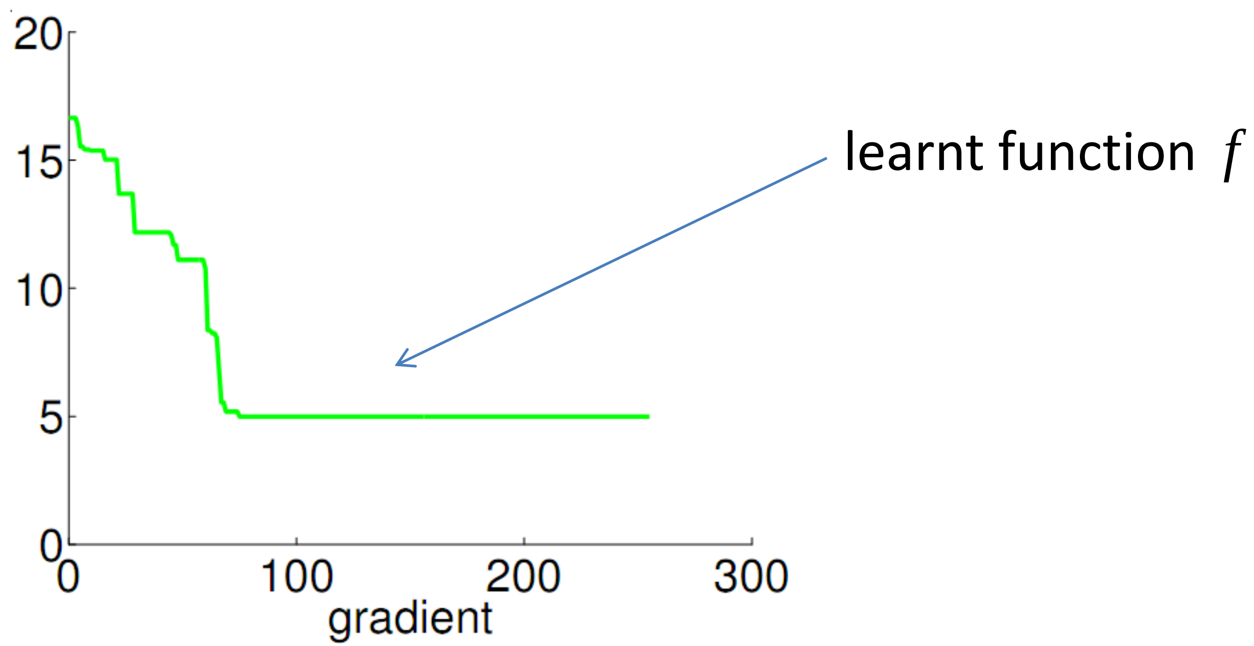

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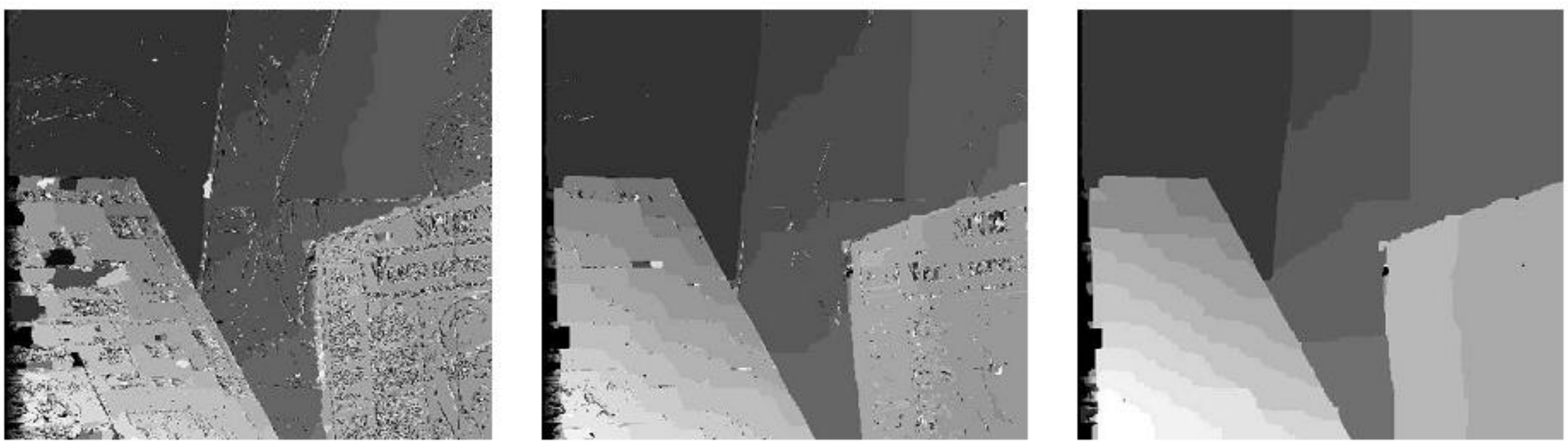

“Venus” disparity using $f(\cdot)$ as estimated at different iterations of learning algorithm

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Sawtooth 4.9% | Poster 3.7% | Bull 2.8%

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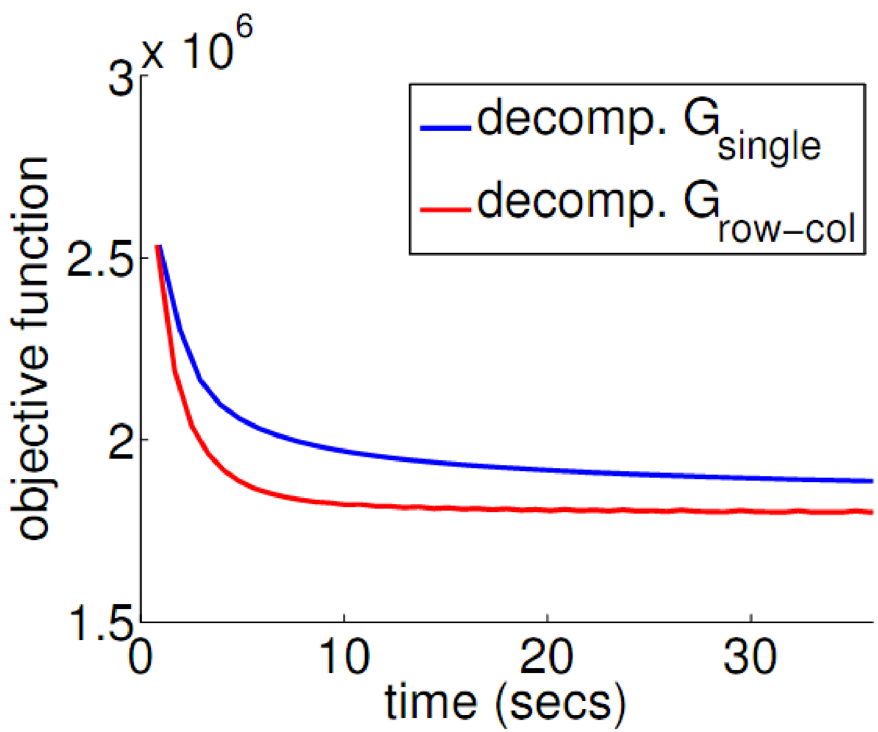

High-order P^n Potts model

Goal: learn high order CRF with potentials given by

$$h_c(\mathbf{x}) = \begin{cases} \beta_l^c & \text{if } x_p = l,\ \forall p \in c \\ \beta_{\max}^c & \text{otherwise} \end{cases},$$

[Kohli et al. CVPR07]

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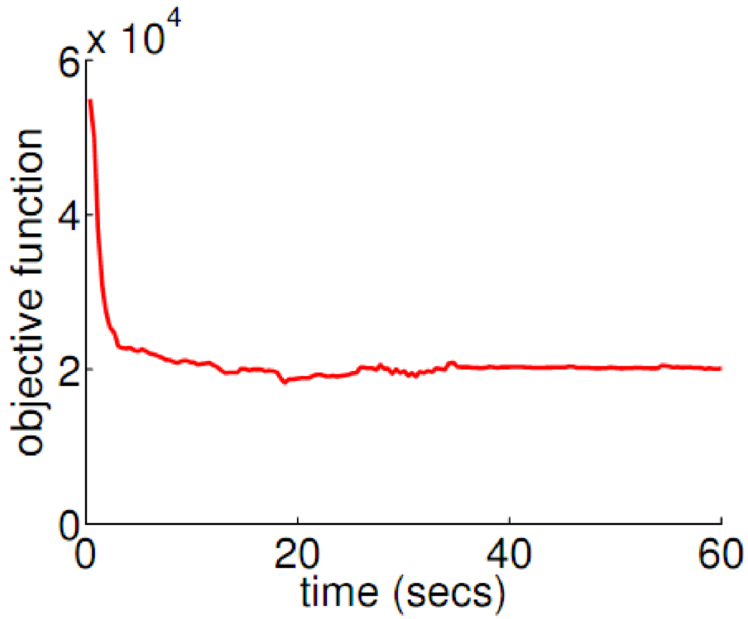

- 100 training samples
- 50x50 grid
- clique size 3x3
- 5 labels (|L|=5)

Learning to cluster [ICCV 2011]

Clustering

- A fundamental task in vision and beyond
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- **Goal 1:** learn this distance automatically based on training data
- **Goal 2:** learning should also handle the fact that the number of clusters is typically unknown at test time

Exemplar based clustering formulation

$$\min_{Q \subseteq S} E(Q) = \sum_{p \notin Q} \min_{q \in Q} d_{p,q} + \sum_{q \in Q} d_{q,q}$$

set of datapoints

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Inference can be performed efficiently using:
Clustering via LP-based Stabilities [Komodakis et al., NIPS 2008]

Exemplar based clustering as a high-order CRF

$$
\begin{aligned}
\min_{\mathbf{x}} \sum_{p,q \in S} & d_{p,q} x_{pq} \\
\text{s.t.} \sum_{q \in S} & x_{pq} = 1, \ \ \forall p \\
& x_{pq} \leq x_{qq}, \qquad \forall p, q \\
& x_{pq} \in \{0, 1\}, \ \ \forall p, q.
\end{aligned}
$$

Exemplar based clustering as a high-order CRF

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$$E(\mathbf{x}; \mathbf{d}) = \sum_{p,q} \underbrace{d_{p,q} x_{pq}}_{\text{unary terms}} + \sum_{p,q} \underbrace{\delta(x_{pq} \leq x_{qq})}_{\text{pairwise terms}} + \sum_{p} \underbrace{\delta\left(\sum_q x_{pq} = 1\right)}_{\text{higher-order terms}}$$

$$\delta(a) = \begin{cases} 0, & a \text{ is true} \\ \infty, & a \text{ is false} \end{cases}$$

Learning to cluster via high-order latent CRFs

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  - Vector valued feature function $g_{pq}(\cdot)$

$$d_{p,q}^k = \mathbf{w}^T g_{pq}(\mathbf{z}^k)$$

Learning to cluster via high-order latent CRFs

- Loss function for clustering

$$\Delta(\mathbf{x};\mathcal{C}^k)=\alpha\sum_{C\in\mathcal{C}^k}\left|1-\sum_{q\in C}x_{qq}\right|+\beta\sum_{C\in\mathcal{C}^k}\sum_{p\in C}\left(1-\sum_{q\in C}x_{pq}\right)$$

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measures inconsistency between **x** and partition $\mathcal{C}^k$

- Set of clusterings fully consistent with partition C^k

$$\mathcal{X}(\mathcal{C}^k) = \{\mathbf{x} : \Delta(\mathbf{x};\mathcal{C}^k) = 0\}$$

Learning to cluster via high-order latent CRFs

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Learning to cluster via high-order latent CRFs

How to efficiently deal with these problems during learning?

Learning to cluster via high-order latent CRFs

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Solution: CRF training via **dual decomposition** for **latent CRFs**

Learning to cluster via high-order latent CRFs

$$\bar{E}^k(\mathbf{x}; \mathbf{w}) := E(\mathbf{x}; \mathbf{d^k}) - \Delta(\mathbf{x}; \mathcal{C}^k)$$

Learning to cluster via high-order latent CRFs

$$\bar{E}^k(\mathbf{x};\mathbf{w}) = \sum_{p,q} \bar{u}^k_{pq}(x_{pq}) + \sum_{p,q} \bar{\phi}_{pq}(x_{pq}, x_{qq}) + \sum_{p} \bar{\phi}_p(\mathbf{x}_p) + \sum_{C \in \mathcal{C}^k} \bar{\phi}_C(\mathbf{x}_C) - \beta |S^k|$$

Learning to cluster via high-order latent CRFs

$$\boxed{\left(d_{p,q} + \beta\cdot\left[\exists C \in \mathcal{C}^k : p, q \in C\right]\right)\cdot x_{pq}}$$

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Choosing decomposition for clustering

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One slave CRF per ground truth cluster C

$$\bar{E}^k_C(\mathbf{x};\mathbf{w},\boldsymbol{\theta}^k) = \sum_{q\in C} \theta^k_{Cq} x_{qq} + \bar{\phi}_C(\mathbf{x}_C)$$

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Learning to cluster via high-order latent CRFs

$$\min_{\{\mathbf{x}^k \in \mathcal{X}(\mathcal{C}^k)\}, \mathbf{w}, \{\boldsymbol{\theta}^k \in \boldsymbol{\Theta}^k\}} R(\mathbf{w}) + \sum_k \sum_{p \in S^k} \mathcal{L}_{\bar{E}_p^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) + \sum_k \sum_{C \in \mathcal{C}^k} \mathcal{L}_{\bar{E}_C^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k)$$

$$\mathcal{L}_{\bar{E}}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) := \bar{E}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) - \min_{\mathbf{x}} \bar{E}(\mathbf{x}; \mathbf{w}, \boldsymbol{\theta}^k)$$

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- Use block coordinate descent
- Alternately optimize
  a. $\{\mathbf{x}^k\}$
  b. $\left\{\mathbf{w}, \{\boldsymbol{\theta}^k \in \boldsymbol{\Theta}^k\}\right\}$

Optimizing over $\{\mathbf{x}^k\}$

$$\min_{\{\mathbf{x}^k \in \mathcal{X}(\mathcal{C}^k)\}, \cancel{\mathbf{w}, \{\boldsymbol{\theta}^k \in \Theta^k\}}} R(\mathbf{w}) + \sum_k \sum_{p \in S^k} \mathcal{L}_{\bar{E}_p^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) + \sum_k \sum_{C \in \mathcal{C}^k} \mathcal{L}_{\bar{E}_C^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k)$$

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$$\min_{\{\mathbf{x}^k \in \mathcal{X}(\mathcal{C}^k)\}, \mathbf{w}, \cancel{\{\boldsymbol{\theta}^k \in \Theta^k\}}} R(\mathbf{w}) + \sum_k \sum_{p \in S^k} \mathcal{L}_{\bar{E}_p^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) + \sum_k \sum_{C \in \mathcal{C}^k} \mathcal{L}_{\bar{E}_C^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k)$$

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optimal cluster centers (exemplars)

$$Q^k = \{q_C\}_{C \in \mathcal{C}^k} \qquad q_C = \arg\min_{q \in C} \sum_{p \in C} d_{p,q}^k$$

Optimizing over $\{\mathbf{x}^k\}$

$$\min_{\{\mathbf{x}^k \in \mathcal{X}(\mathcal{C}^k)\}, \mathbf{w}, \cancel{\{\boldsymbol{\theta}^k \in \Theta^k\}}} R(\mathbf{w}) + \sum_k \sum_{p \in S^k} \mathcal{L}_{\bar{E}_p^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) + \sum_k \sum_{C \in \mathcal{C}^k} \mathcal{L}_{\bar{E}_C^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k)$$

$$\mathcal{L}_{\bar{E}}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) := \bar{E}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) - \min_{\mathbf{x}} \bar{E}(\mathbf{x}; \mathbf{w}, \boldsymbol{\theta}^k)$$

$$\mathbf{x}^k = \arg \min_{\mathbf{x} \in \mathcal{X}(\mathcal{C}^k)} E(\mathbf{x}; \mathbf{d}^k)$$

optimal cluster centers (exemplars)

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Optimizing over $\{\mathbf{w}, \{\boldsymbol{\theta}^k \in \boldsymbol{\Theta}^k\}\}$

$$\min_{\cancel{\{\mathbf{x}^k \in \mathcal{X}(\mathcal{C}^k)\}}, \mathbf{w}, \{\boldsymbol{\theta}^k \in \boldsymbol{\Theta}^k\}} R(\mathbf{w}) + \sum_k \sum_{p \in S^k} \mathcal{L}_{\bar{E}_p^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k) + \sum_k \sum_{C \in \mathcal{C}^k} \mathcal{L}_{\bar{E}_C^k}(\mathbf{x}^k; \mathbf{w}, \boldsymbol{\theta}^k)$$

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$\{\mathbf{x}^k\}$ is known

- Back to **fully supervised** learning

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- As already explained, in this case training requires solving the slave CRFs

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Solving slave CRF $\bar{E}_C^k$

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$$\Downarrow$$

$$\forall q \in C,\ \hat{x}_{qq} = \begin{cases} [\theta_q^k < \alpha], & \mathit{if}\ 2\alpha + \sum_{q'\in C}[\theta_{q'}^k - \alpha]_- < 0 \\ 0, & \mathit{otherwise} \end{cases}$$

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$$\forall q \neq p,\ \hat{x}_{qq} \leftarrow [\theta_q^k < 0]$$

$$\forall q,\ \hat{x}_{pq} \leftarrow [q = \bar{q}],\ \textit{where}\ \bar{q} = \arg\min_q \bar{\theta}_q^k$$

Learning scheme

Data: training samples $\{\mathcal{C}^k, \mathbf{z}^k\}_{k=1}^K$, features $\{f_{pq}(\cdot)\}$

repeat

/* Optimize over $\mathbf{x}^k$ */

compute optimal set of exemplars Q^k

set $x_{qq}^k = 1 \Leftrightarrow q \in Q^k, \quad x_{pq}^k = 1 \Leftrightarrow q = \arg\min_{q \in Q^k} d_{p,q}^k, \ \forall p \neq q$

/* Apply T rounds of projected subgradient */

repeat T times {

get solutions $\hat{\mathbf{x}}^{k,p}$, $\hat{\mathbf{x}}^{k,C}$ of slaves $\bar{E}_p^k$, $\bar{E}_C^k$ to estimate subgradient

update $\mathbf{w}, \boldsymbol{\theta}^k$ via projected subgradient update

}

until convergence

Training high-order latent CRFs via dual decomposition

- More generally, dual decomposition can be used for training any **high-order latent** model

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$$h_c^k\big((\mathbf{x}_c, \tilde{\mathbf{x}}_c)\big) = \mathbf{w}^T g_c((\mathbf{x}_c, \tilde{\mathbf{x}}_c), \mathbf{z}^k)$$

vector valued feature functions

Learning a weighted Euclidean distance

- We consider a weighted Euclidean distance d_{pq} for D-dimensional datapoints

$$d_{pq} = \sum_{i=1}^{D} w_i (x_p^i - x_q^i)^2$$

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- **Goal:** learn weights w_i automatically from clustering data

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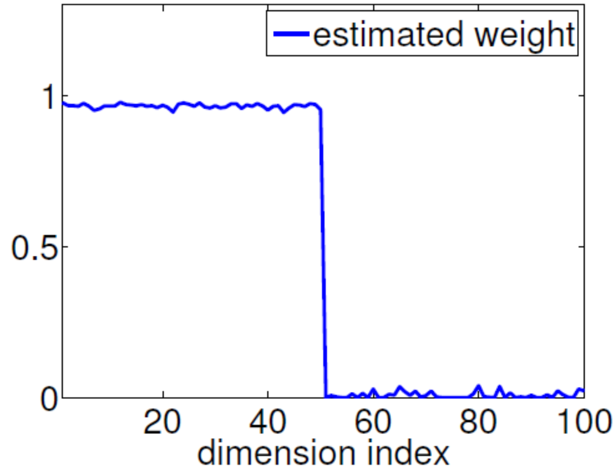

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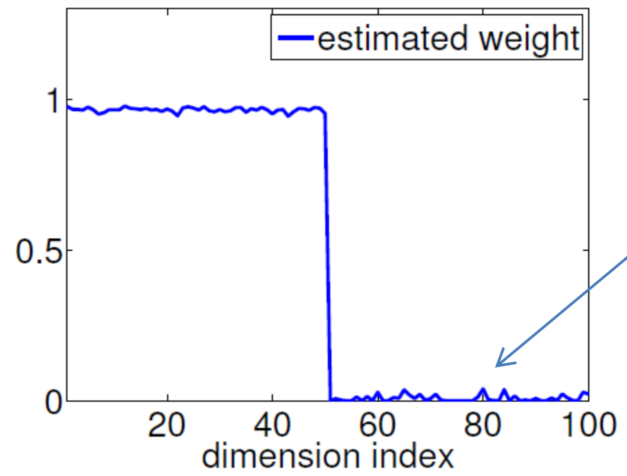

noisy dimensions get suppressed weights after training

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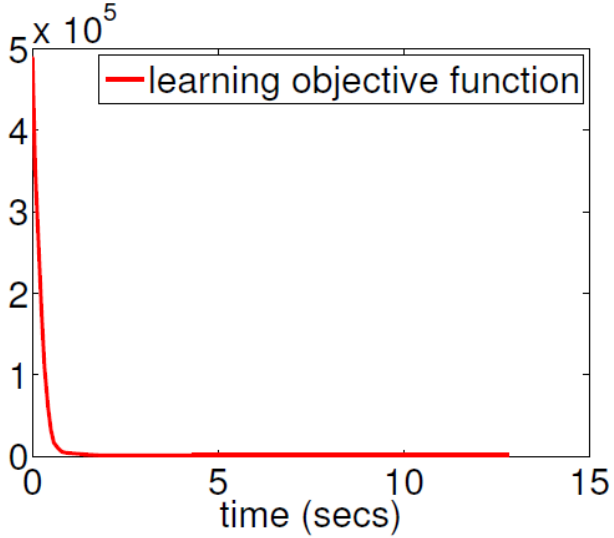

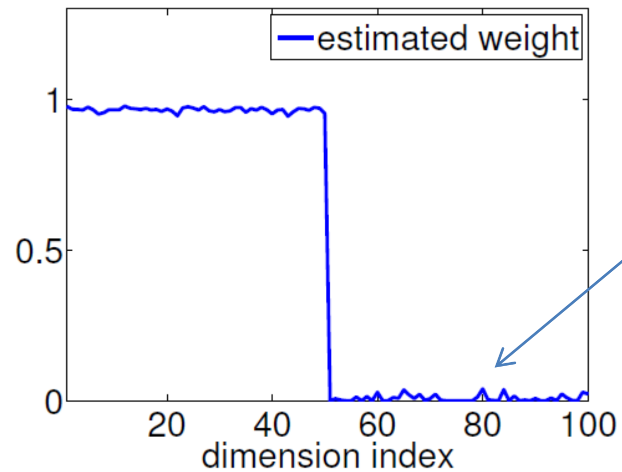

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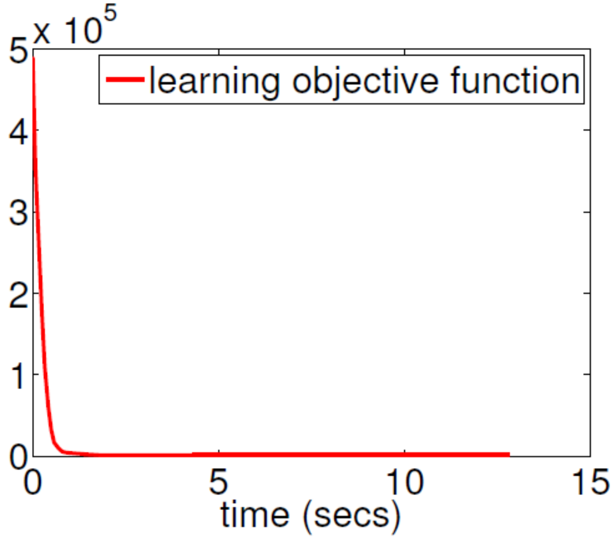

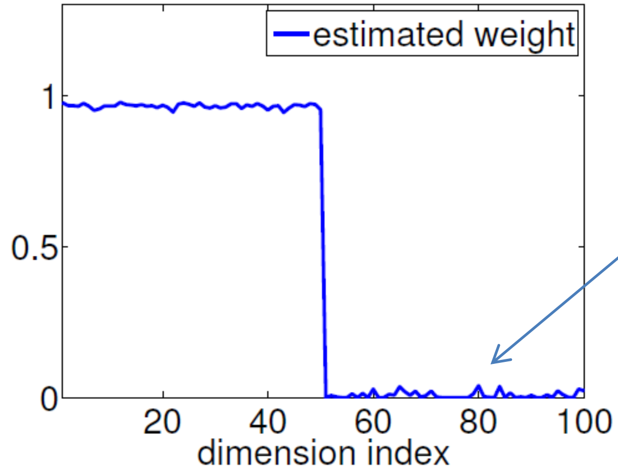

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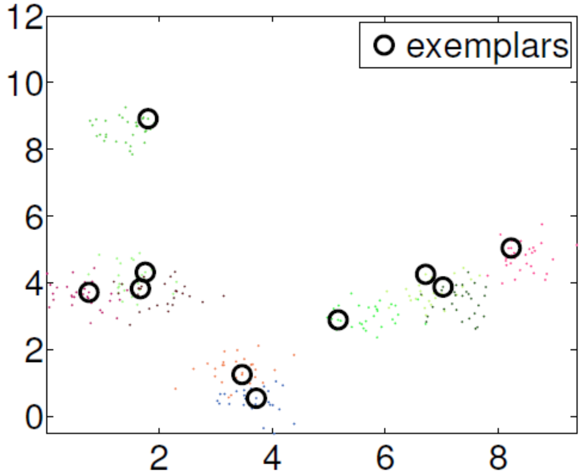

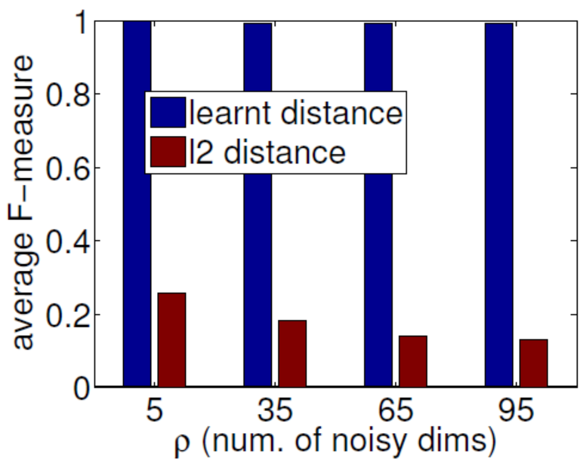

Learning to cluster texture images

Learn weighted comb. of distances between features: $d(\cdot) = \sum_f w_f d^f(\cdot)$

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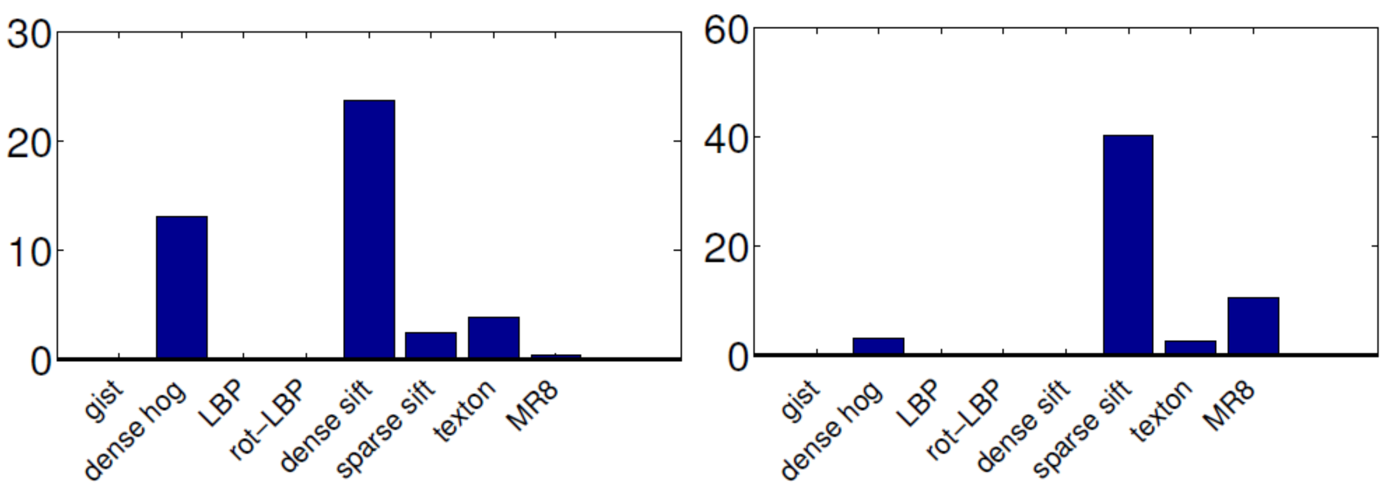

(a) Outex (b) UIUC

clustering accuracy:

- 100% (Outex)
- 86% (UIUC)

learnt weights

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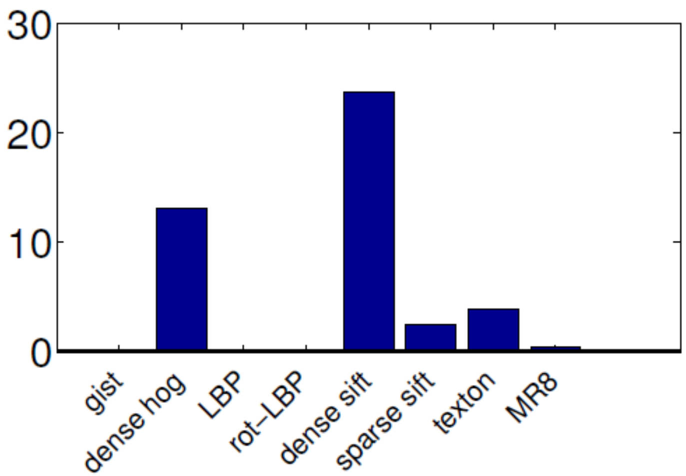

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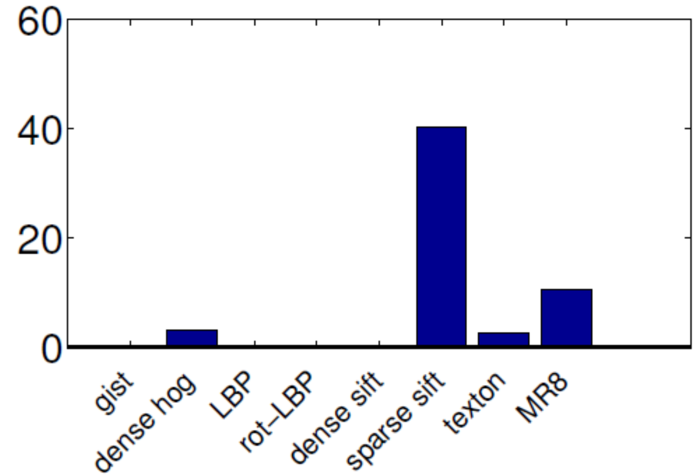

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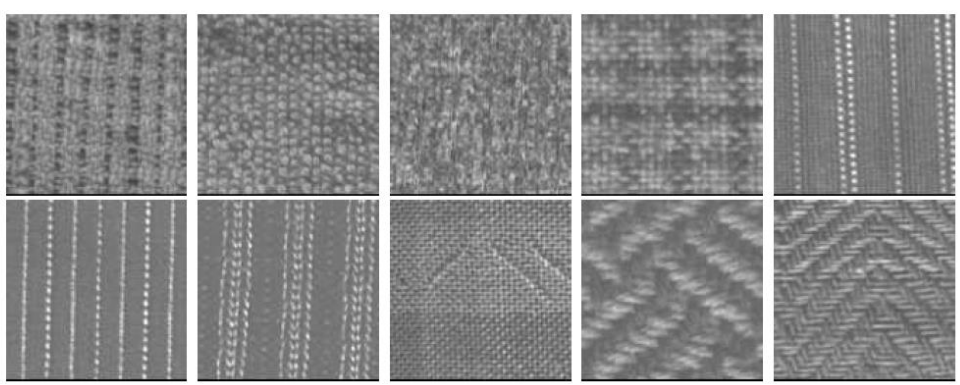

10 of the estimated exemplars for Outex

Learning to cluster scene images

Learn weighted combination of distances (multiple distances per feature, multiple features)

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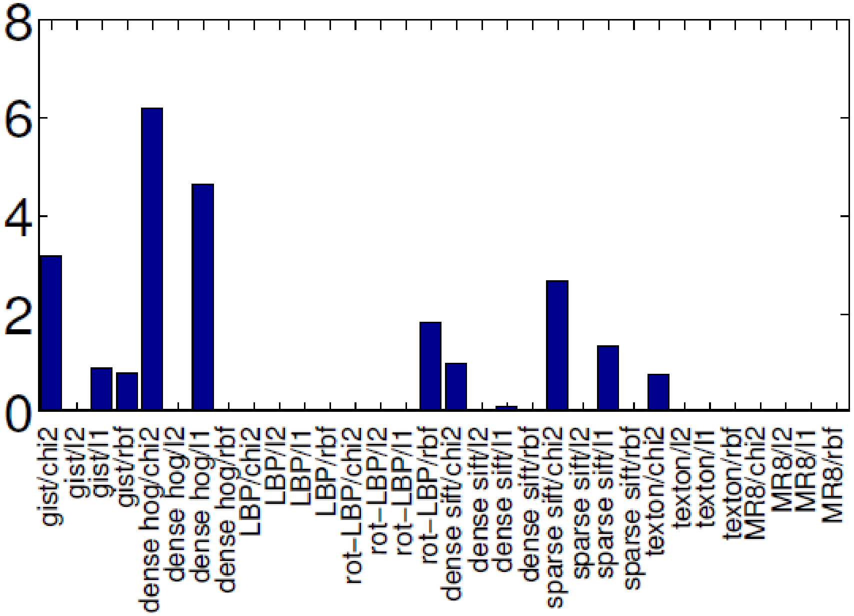

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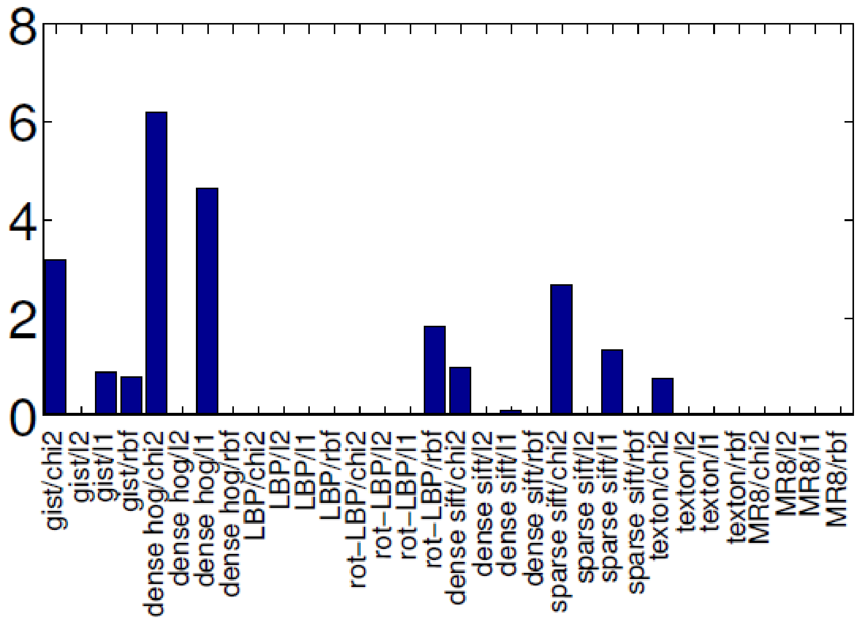

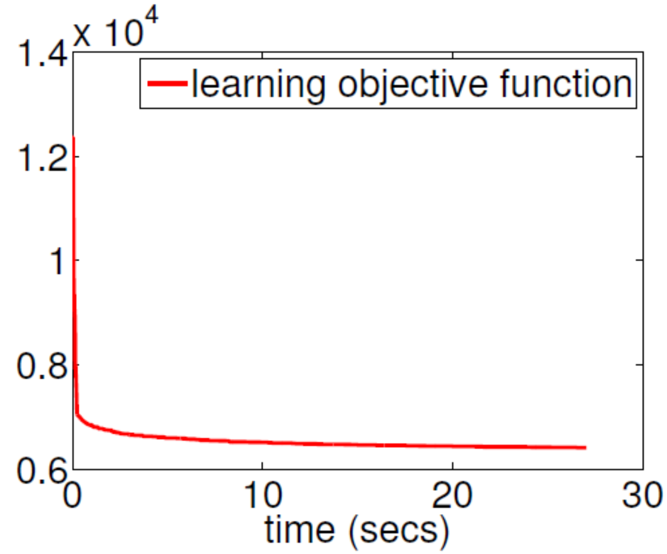

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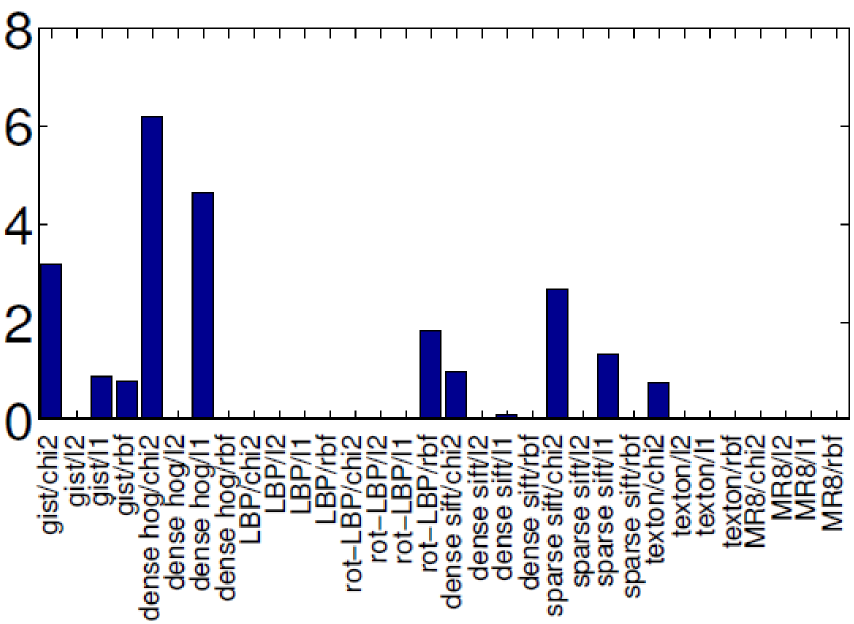

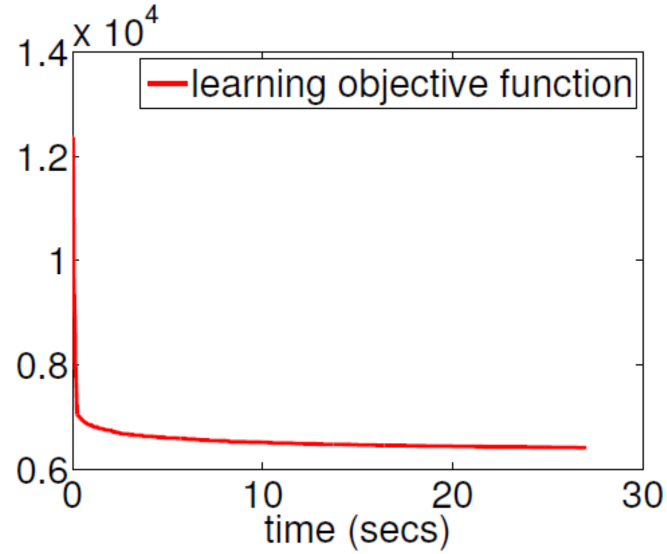

clustering accuracy: 63% (Scene)

10 of the estimated exemplars for Outex

Thank you for your attention!

Questions?