Multi-label energy minimization
via graph cuts
Stereo matching

- Extract correspondences between similar images
- Images are typically assumed to be horizontally aligned (rectification)
Correspondences via horizontal shifts (called “disparities”)
Correspondences via horizontal shifts (called “disparities”)
Visualization of disparities (disparity map)

Disparity inversely proportional to depth
Window based approach

- Winner-takes-all approach
- Windows matched independently
- Small or large windows can be used
- With a simple trick, running time can be made independent of window size
Small vs large windows

small window
- better at boundaries
- noisy in low texture areas

large window
- better in low texture areas
- blurred boundaries
Scan-line approaches

- Match scan lines independently, i.e., introduce coherence only along scanlines (what is the resulting MRF?)

- Better than window-based approach

- But still not good enough

- Problem: streaking artifacts
Graph-cut approach

- We will use a 2D grid for our MRF

- We will penalize disparity discontinuities both in horizontal or vertical direction

- Much better modeling (spatial coherence along AND across scanlines)
Graph-cut approach

- Resulting MRF energy:

\[ \sum_{p} (I_p - I_{p+d_p})^2 + \sum_{pq} w_{pq} |d_p - d_q| \]

- Photo-consistency
- Spatial coherence

- How can we select the weights \( w_{pq} \)?

- Why not just apply loopy-BP in this case?
MRF optimization via graph-cuts

- Optimizing MRF energies of the following form:

\[ \sum_p V_p(d_p) + \sum_{pq} w_{pq} |d_p - d_q| \]

- Belief propagation can not guarantee an optimal solution (loopy graph)

- We will use graph-cut based methods (exact global optimum in polynomial time)

- But how can this be reduced to a graph-cut problem?
MRF optimization via graph-cuts
MRF optimization via graph-cuts

Disparity labels

labels

L(p)

"cut"

p
Let’s concentrate on one pair of neighboring pixels \((p,q)\)

\[
E(d_p, d_q) = D_p(2) + D_q(5) + \ldots + w_{pq} \cdot \left| 3 \right| + \ldots
\]

**Disparity labels**

- -5
- -4
- -3
- -2
- -1
- 0

**Cost of vertical edges**

**Cost of horizontal edges**
MRF optimization via graph-cuts

Let’s concentrate on one pair of neighboring pixels $(p, q)$

Disparity labels

\[ E(d_p, d_q) = D_p(d_p) + D_q(d_q) + \ldots \]

Cost of vertical edges

\[ + w_{pq} \cdot |d_p - d_q| + \ldots \]

Cost of horizontal edges
MRF optimization via graph-cuts

The combined energy over the entire grid $G$ is:

$$E(d) = \sum_{p \in G} D_p(d_p)$$

$$+ \sum_{\{p,q\} \in N} w_{pq} \cdot |d_p - d_q|$$

**(photo consistency)**

(cost of vertical edges)

**(spatial consistency)**

(cost of horizontal edges)
Scan-line stereo vs. Multi-scan-line stereo

Dynamic Programming
(single scan line optimization)

s-t Graph Cuts
(multi-scan-line optimization)
Scan-line vs. graph-cut stereo

multi scan line stereo (graph cuts)  single scan-line stereo (DP)
Scan-line vs. graph-cut stereo

multi scan line stereo (graph cuts)  single scan-line stereo (DP)
Multi-label energy minimization via graph-cuts (continued)
Optimizing multi-label MRFs via graph-cuts

- We saw how to use graph cuts to optimize:
  - binary MRFs
  - restricted class of multi-label MRFs

- Next
  - Using graph-cuts to optimize a more general class of MRFs
  - More general refers to the pairwise potentials
Optimizing multi-label MRFs via graph-cuts

- Linear interactions
  - Can be optimized exactly using graph-cuts
  - We saw graph construction in previous class

- Actually a similar graph construction applies to “convex” interactions
  (sketch graph construction on board)
Why more general pairwise interactions?

- “convex” vs “discontinuity preserving” interactions

```
V(dx) = \begin{cases} 
(\delta p - \delta q) & \text{if } dx = x_p - x_q \\
0 & \text{otherwise} 
\end{cases}
```

“linear” model

```
V(dx) = \begin{cases} 
\delta p - \delta q & \text{if } dx = x_p - x_q \\
0 & \text{otherwise} 
\end{cases}
```

Potts model

```
V(dx) = \begin{cases} 
\delta p - \delta q - \delta_k & \text{if } dx = x_p - x_q \\
\delta_k & \text{otherwise} 
\end{cases}
```
Why more general pairwise interactions?

“linear” $V$

truncated “linear” $V$
MRF hardness

- Hardness of optimizing an MRF depends critically on the type of its pairwise potentials (i.e. the distance function between labels)
Handling more general MRFs via graph-cuts

- We will restrict the type of pairwise potentials
  - Pairwise potential is assumed to be a metric distance between labels (metric potential)

- We will derive an algorithm called alpha-expansion

- Approximate optimality guarantees
Alpha-expansion

- Idea: try to optimize multi-label MRF using binary MRFs

- We will have to use many binary MRFs

- Iterative algorithm
  - Binary MRF optimized at each step

- Improving our solution incrementally
As its name reveals, alpha-expansion relies on the so-called expansion moves.
Alpha-expansion

- Optimal expansion
  - Given any label $a$, the optimal $a$-expansion move is the one that yields the minimum energy

- Given label $a$, how can we find the optimal $a$-expansion move with respect to the current configuration?

- What does this problem reduce to? (show on board)
Alpha-expansion for multi-label MRF

1. Start with any labeling

2. run through all labels and for each label $a$
   2a. compute optimal $a$-expansion move
   2b. if better energy found, accept the move

3. Stop if no label change, otherwise goto 2
Alpha-expansion for multi-label MRF

1. Start with any labeling

2. run through all labels and for each label a
   2a. Binary MRF optimization
   2b. if better energy found, accept the move

3. Stop if no label change, otherwise goto 2
Alpha-expansion example

For each move we choose expansion that gives the largest decrease in the energy: **binary optimization problem**
Alpha-expansion

- How do we optimize the binary MRFs encountered during alpha-expansion?

- Well, we will reduce them to graph-cut problems (we already know how to do that for certain binary MRFs)
Alpha-expansion for multi-label MRF

1. Start with any labeling

2. run through all labels and for each label a
   2a. compute optimal a-expansion move
   2b. if better energy found, accept the move

3. Stop if no label change, otherwise goto 2
Alpha-expansion for multi-label MRF

1. Start with any labeling

2. run through all labels and for each label \( a \)
   2a. Binary MRF optimization
   2b. if better energy found, accept the move

3. Stop if no label change, otherwise goto 2
Alpha-expansion for multi-label MRF

1. Start with any labeling

2. run through all labels and for each label a
   2a. s-t graph cut problem
   2b. if better energy found, accept the move

3. Stop if no label change, otherwise goto 2

alpha-expansion thus reduces to solving a series of graph-cut problems