MVA/IMA – 3D Vision

Graph Cuts and Application to Disparity Map Estimation

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(with many borrowings from Boykov & Veksler 2006)

Introduction

3D reconstruction

- capturing reality
  - for diagnosis, simulation, movies, video games, interaction in virtual/augmented reality, ...

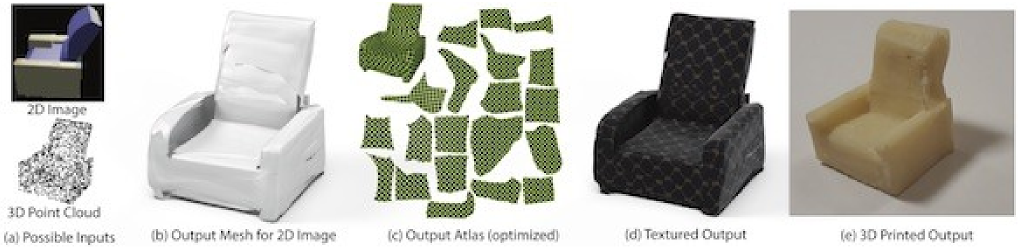
This course:

- camera calibration
  - relevance of accuracy: 1° error, at 10m 17cm error
- low-level 3D (disparity/depth map, mesh)
  - as opposed to high-level geometric primitives, semantics...

Mathematical tools for 3D reconstruction

- Deep learning:
  - very good for matching image regions

    → subcomponent of 3D reconstruction algorithm
  - a few methods for direct disparity/depth map estimation
  - fair results on 3D reconstruction from single view



Groueix et al. 2017

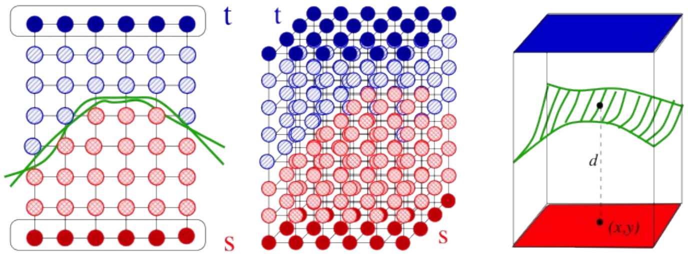
- Graph cuts (this lecture):
  - practical, well-founded, general (→ maps, meshes...)

Motivating graph cuts

- Powerful **multidimensional energy minimization** tool
  - wide class of binary and non binary energies
  - in some cases, globally optimal solutions
  - some provably good approximations (and good in practice)
  - allowing regularizers with contrast preservation
    - enforcement of piecewise smoothness while preserving relevant sharp discontinuities

$$E(f) = \sum_{p \in \mathrm{P}} D_p(f_p) + \sum_{(p,q) \in \mathrm{N}} V_{p,q}(f_p, f_q)$$

- Geometric interpretation
  - hypersurface in *n*-D space



Many links to other domains

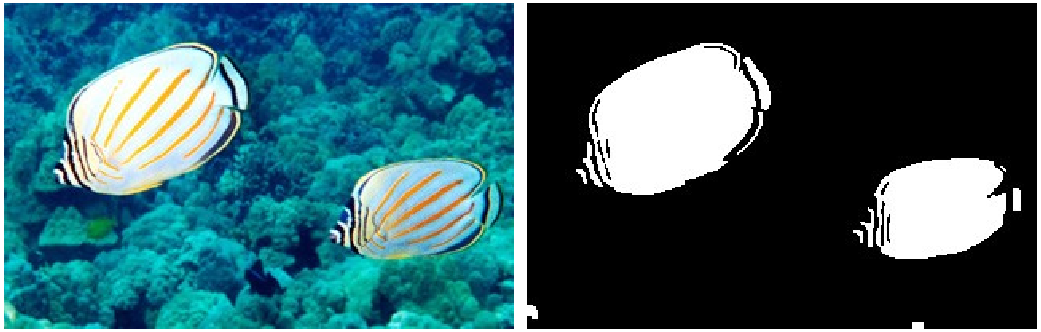
(cf. Boykov & Veksler 2006)

- Combinatorial algorithms (e.g., dynamic programming)
- Simulated annealing
- Markov random fields (MRFs)
- Random walks and electric circuit theory
- Bayesian networks and belief propagation
- Level sets and other variational methods
- Anisotropic diffusion
- Statistical physics
- Submodular functions
- Integral/differential geometry, etc.

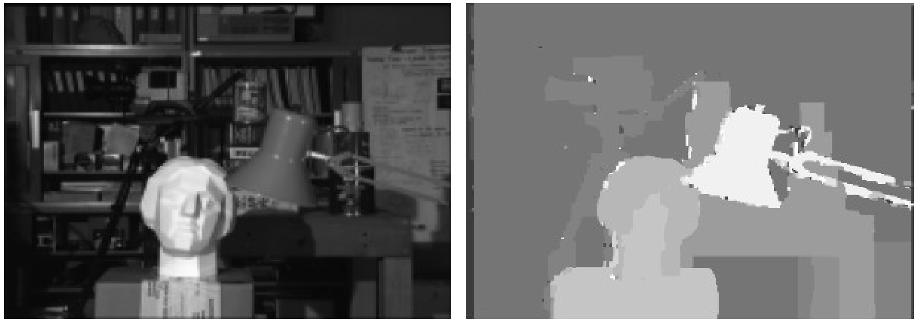
dynamic programming = programmation dynamique
simulated annealing = recuit simulé
Markov random field = champ (aléatoire) de Markov
random walk = marche aléatoire
Bayesian network = réseaux bayésien
level set = ligne de niveau
submodular function = fonction sous-modulaire

Overview of the course

- Notions
  - graph cut, minimum cut
  - flow network, maximum flow
  - optimization: exact (global), approximate (local)
- Illustration with emblematic applications



segmentation

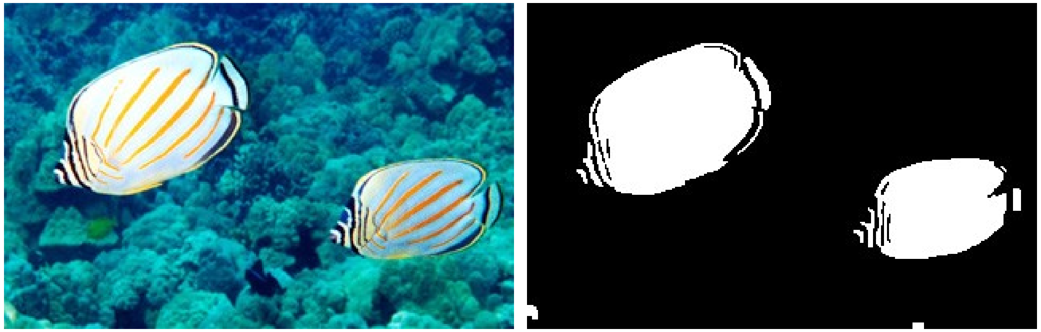


disparity map estimation

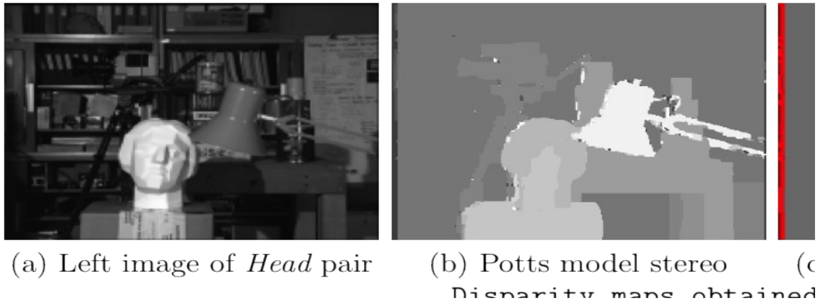
Overview of the course

- Notions
  - graph cut, minimum cut
  - flow network, maximum flow
  - optimization: exact (global), approximate (local)
- Illustration with emblematic applications

No time to go deep
into every topic →
general ideas,
read the references



segmentation



disparity map estimation

Part 1

Graph cuts basics

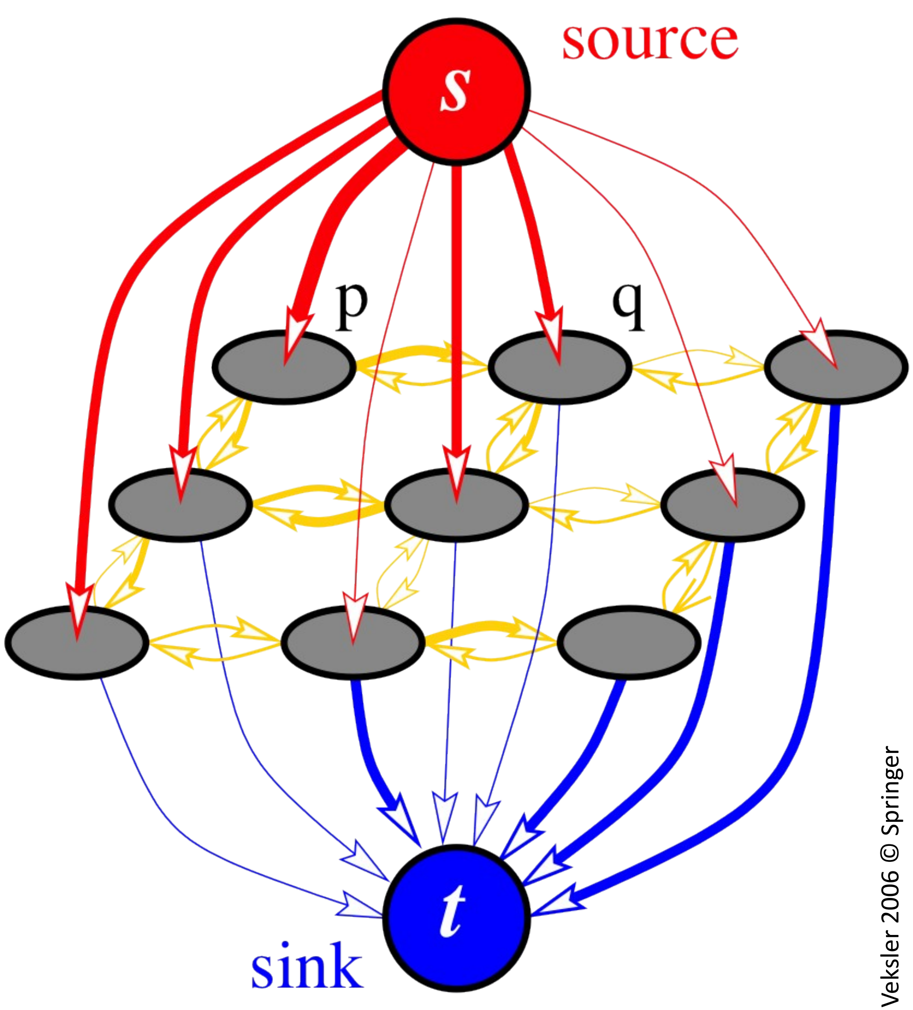
Max-flow min-cut theorem

Application to image restoration and image segmentation

Graph cut basics

node = nœud
vertex (vertices) = sommet(s)
edge = arête
directed = orienté
digraph (directed graph) = graphe orienté
sink = puits

- Graph G = ⟨V,E⟩ (digraph)
  - set of nodes (vertices) V
  - set of directed edges E
    - $p \rightarrow q$
- V = $\{s,t\}$ ∪ P
  - terminal nodes: $\{s,t\}$
    - s: source node
    - t: target node (= sink)
  - non-terminal nodes: P
    - ex. P = set of pixels, voxels, etc.
      (can be very different from an image)



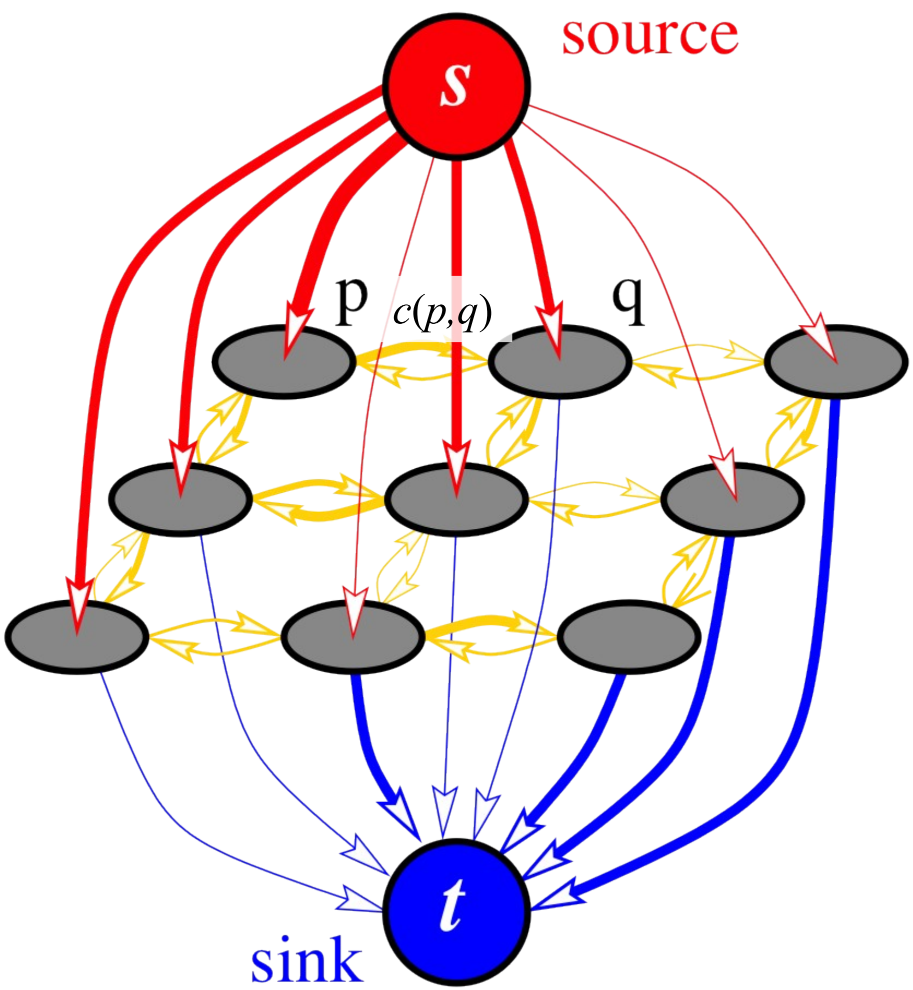
Example of connectivity

Graph cut basics

label = étiquette
weight = poids
link = lien

- Edge labels, for $p \to q \in E$
  - $c(p,q) \geq 0$: **nonnegative** costs

    also called weights $w(p,q)$
  - $c(p,q)$ and $c(q,p)$, <u>if any</u>, may differ
- Links
  - t-link: term. $\leftrightarrow$ non-term.
    - $\{s \to p \mid p \neq t\}$, $\{q \to t \mid q \neq s\}$
  - n-link: non-term. $\to$ non-term.
    - $N = \{p \to q \mid p,q \neq s,t\}$



Boykov & Veksler 2006 © Springer

Cut and minimum cut

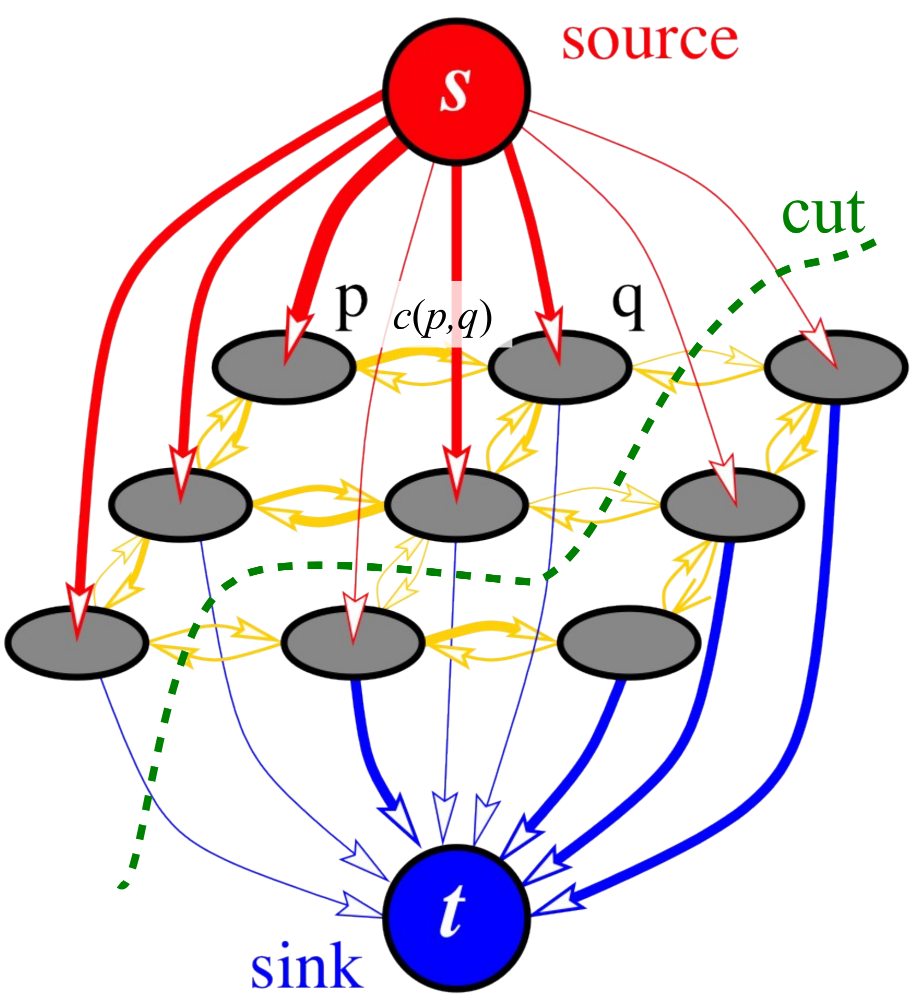
cut = coupe
severed = coupé, sectionné

- ***s-t* cut** (or just "cut"): $C = \{S,T\}$

  node partition such that $s \in S,\ t \in T$

- **Cost of a cut** $\{S,T\}$:
  - $c(S,T) = \sum_{p \in S,\, q \in T} c(p,q)$
  - N.B. cost of severed edges: **only from *S* to *T***

- **Minimum cut:**
  - i.e., with min cost: $\min_{S,T}\ c(S,T)$
  - intuition: cuts only "weak" links



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Different view: flow network

(or transportation network)

flow = flot
network = réseau
transportation = transport
vertex = sommet
node = nœud
edge = arête

- Different vocabulary and features
  - graph ↔ network

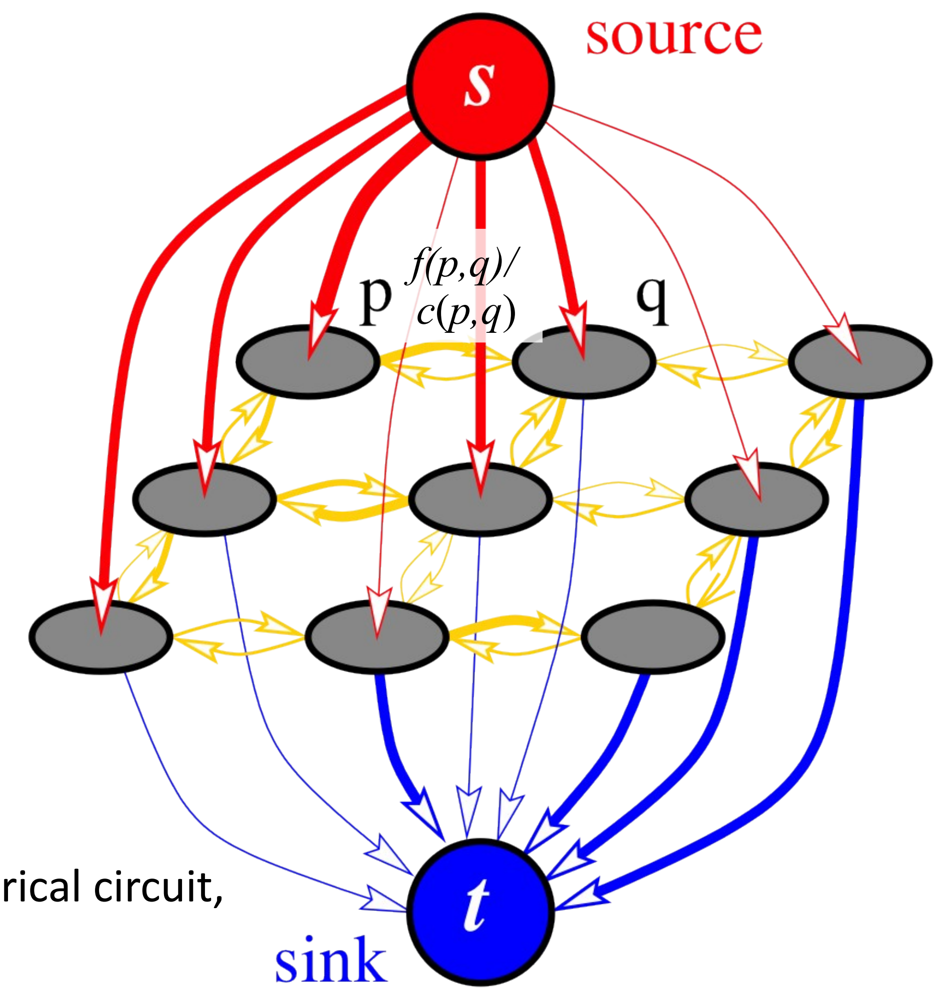
    vertex = node $p, q, \ldots$

    edge = arc $p \rightarrow q$ or (p,q)

    cost = capacity $c(p,q)$

  - possibly many source<u>s</u> & sink<u>s</u>

- Flow $f : V \times V \rightarrow \mathbb{R}$
  - $f(p,q)$: amount of flow $p \rightarrow q$
    - $(p,q) \notin E \Leftrightarrow c(p,q) = 0, f(p,q) = 0$
  - e.g. road traffic, fluid in pipes, current in electrical circuit, …



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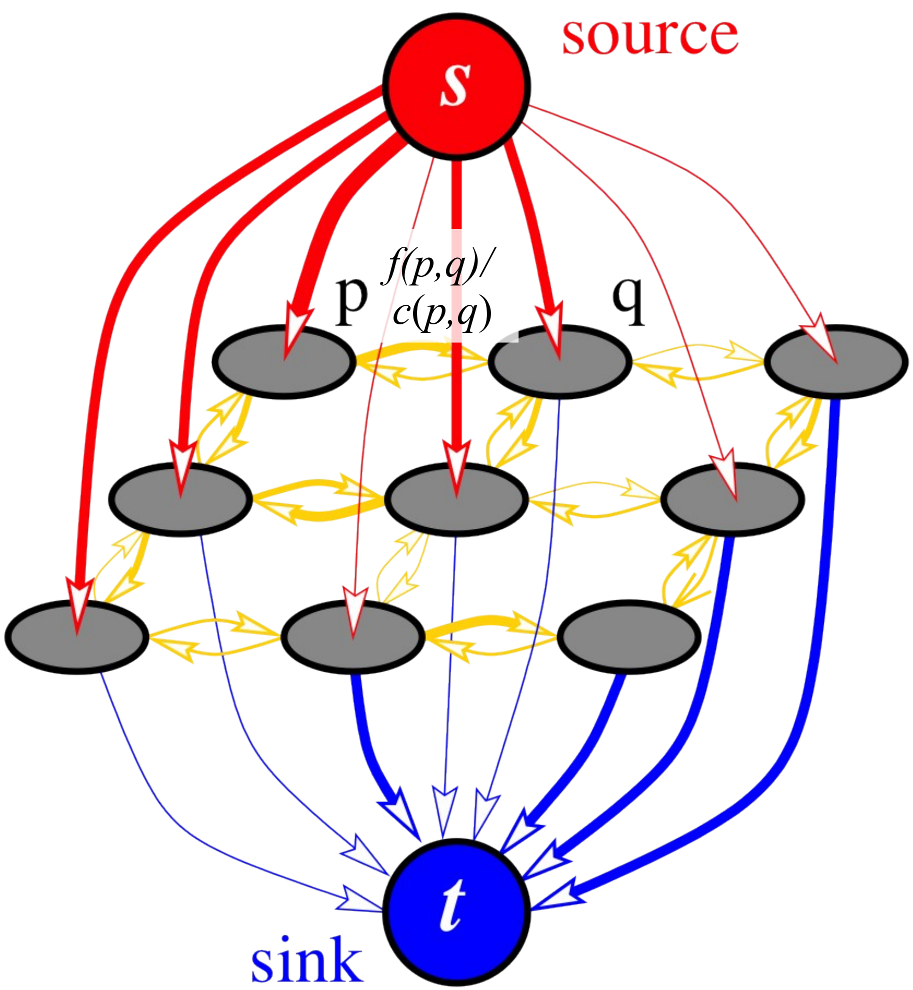
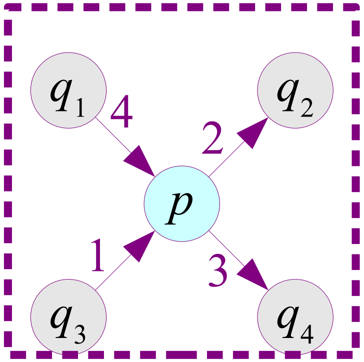
Flow network constraints

skew symmetry = antisymétrie

- Capacity constraint
  - $f(p,q) \leq c(p,q)$
- Skew symmetry
  - $f(p,q) = -f(q,p)$
- Flow conservation
  - $\forall p$, net flow $\sum_{q \in V} f(p,q) = 0$
    unless $p = s$ (s produces flow)
    or $p = t$ (t consumes flow)
  - i.e., incoming $\sum_{(q,p) \in E} f(q,p)$
    $=$ outgoing $\sum_{(p,q) \in E} f(p,q)$





Kirchhoff's law

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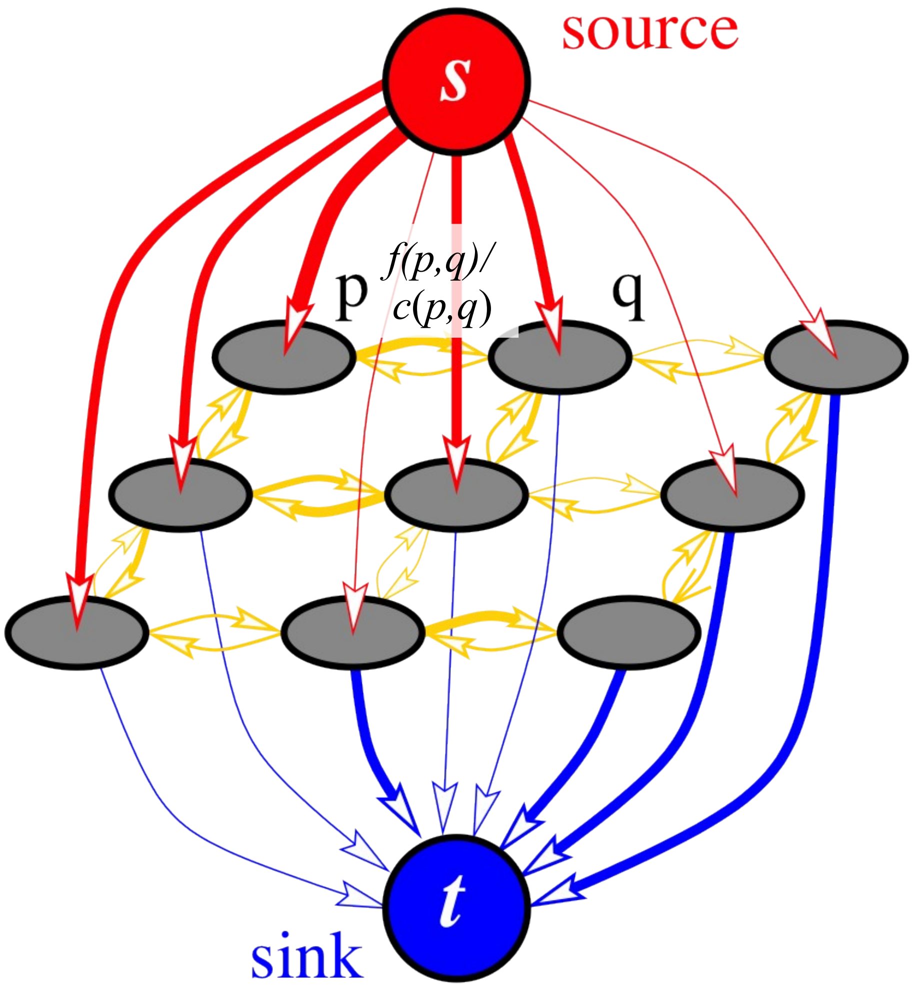
Flow network constraints

skew symmetry = antisymétrie

- ***s-t* flow** (or just "flow) f
  - $f : V \times V \to \mathbb{R}$
    satisfying flow constraints
- **Value of *s-t* flow**
  - $|f| = \sum_{q \in V} f(s,q) = \sum_{p \in V} f(p,t)$
    - amount of flow from source
      = amount of flow to sink
- **Maximum flow:**
  - i.e., with maximum value: $\max_f |f|$
  - intuition: arcs saturated as much as possible



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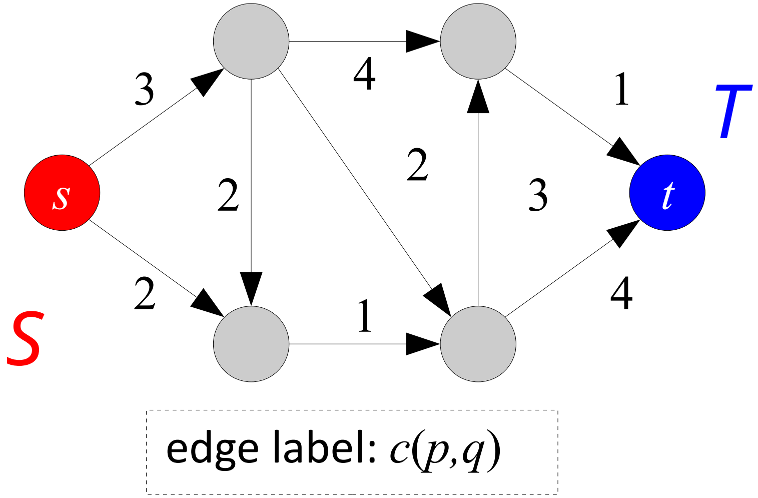
Max-flow min-cut theorem

- Theorem

  The maximum value of an s-t flow is equal to the minimum capacity (i.e., min cost) of an s-t cut.

- Example
  - $|f| = c(\mathrm{S},\mathrm{T}) = ?$



Max-flow min-cut theorem

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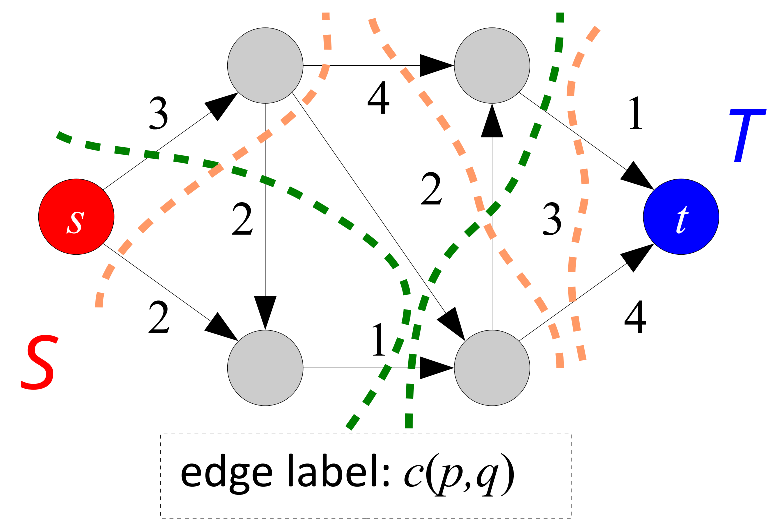
- Example
  - $|f| = c(\mathrm{S},\mathrm{T}) = 4$
  - min: enumerate partitions...



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- Theorem

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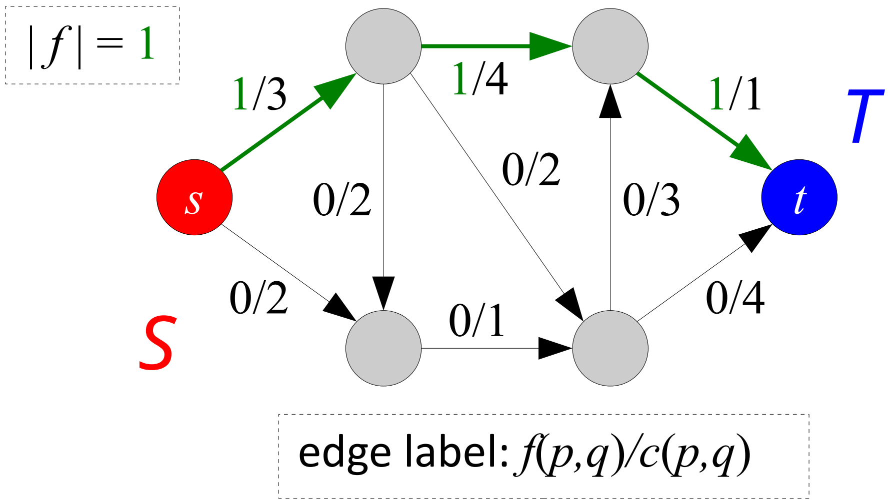
- Example
  - $|f| = c(\mathrm{S},\mathrm{T}) = 4$
  - min: enumerate partitions...
  - max: try increasing $f(p,q)$...



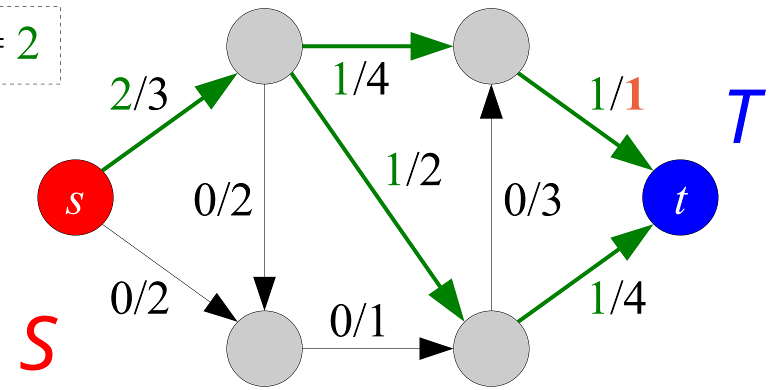
Max-flow min-cut theorem

- Theorem

  The maximum value of an s-t flow is equal to the minimum capacity (i.e., min cost) of an s-t cut.

- Example
  - $|f| = c(\mathrm{S},\mathrm{T}) = 4$
  - min: enumerate partitions...
  - max: try increasing $f(p,q)$...

$|f| = 2$



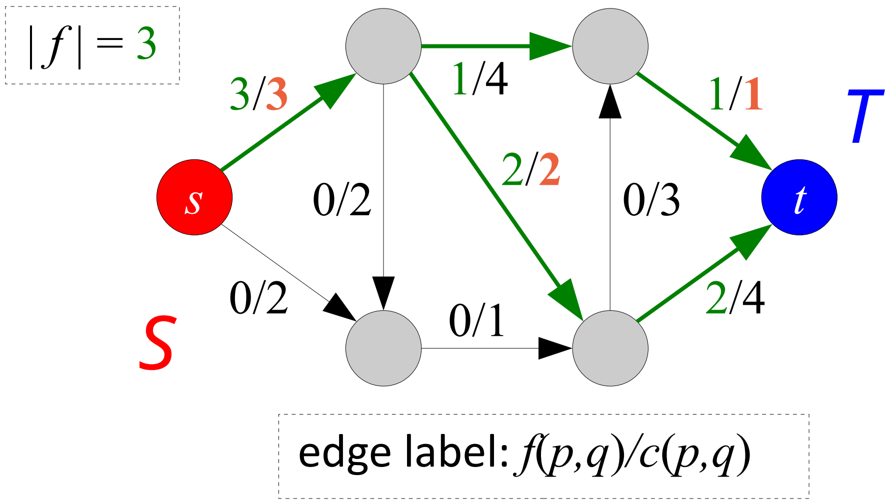
edge label: $f(p,q)/c(p,q)$

Max-flow min-cut theorem

- Theorem

  The maximum value of an s-t flow is equal to the minimum capacity (i.e., min cost) of an s-t cut.

- Example
  - $|f| = c(\mathrm{S},\mathrm{T}) = 4$
  - min: enumerate partitions...
  - max: try increasing $f(p,q)$...

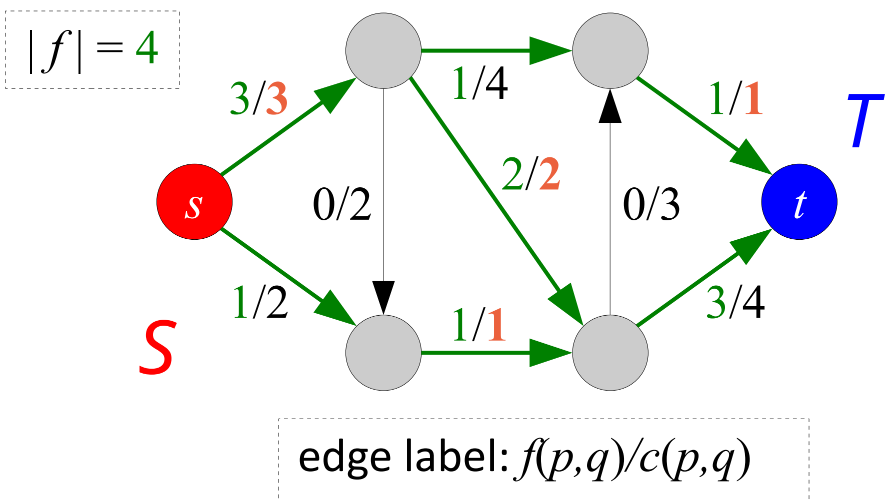


Max-flow min-cut theorem

- Theorem

  The maximum value of an s-t flow is equal to the minimum capacity (i.e., min cost) of an s-t cut.

- Example
  - $|f| = c(\mathrm{S},\mathrm{T}) = 4$
  - min: enumerate partitions...
  - max: try increasing $f(p,q)$...



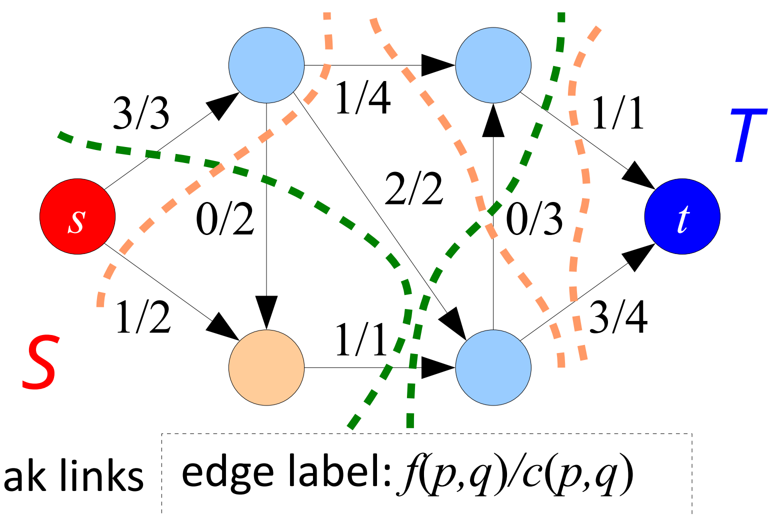
Max-flow min-cut theorem

to pull = tirer
to tear = (se) déchirer
weak = faible

- Theorem

  The maximum value of an s-t flow is equal to the minimum capacity (i.e., min cost) of an s-t cut.

- Example
  - $|f| = c(\mathrm{S,T}) = 4$
  - min: enumerate partitions...
  - max: try increasing $f(p,q)$...



- Intuition
  - pull s and t apart: the graph tears where it is weak
  - min cut: cut corresponding to a small number of weak links
  - max flow: flow bounded by low-capacity links in a cut

Max-flow min-cut theorem

linear programmaing = Programmation linéaire

- Theorem

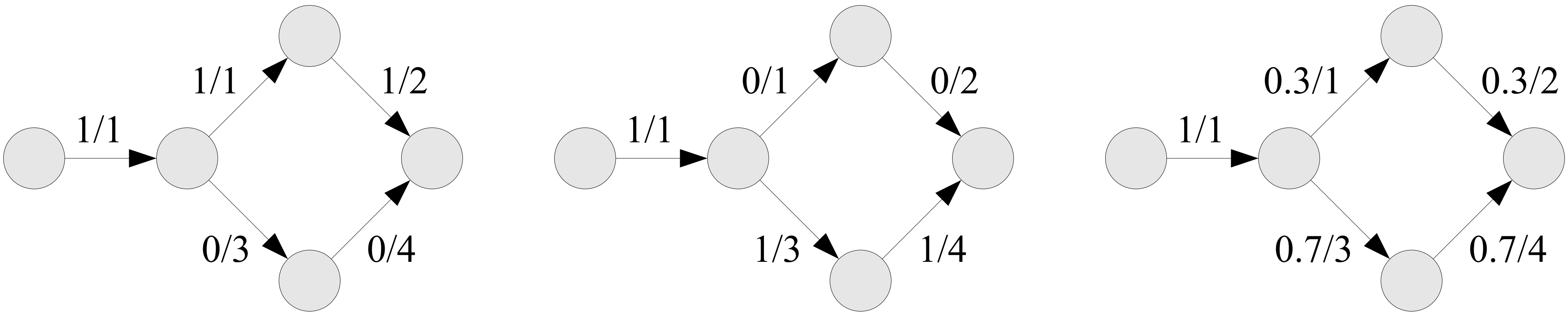
  The maximum value of an s-t flow is equal to the minimum capacity (i.e., min cost) of an s-t cut.

  - proved independently
    by Elias, Feinstein & Shannon,
    and Ford & Fulkerson (1956)
  - special case of strong duality theorem
    in linear programming
  - can be used to derive other theorems

Max flows and min cuts configurations are not unique

- Different configurations with same maximum flow

edge label: $f(p,q)/c(p,q)$



- Different configurations with same min-cut cost

Algorithms for computing max flow

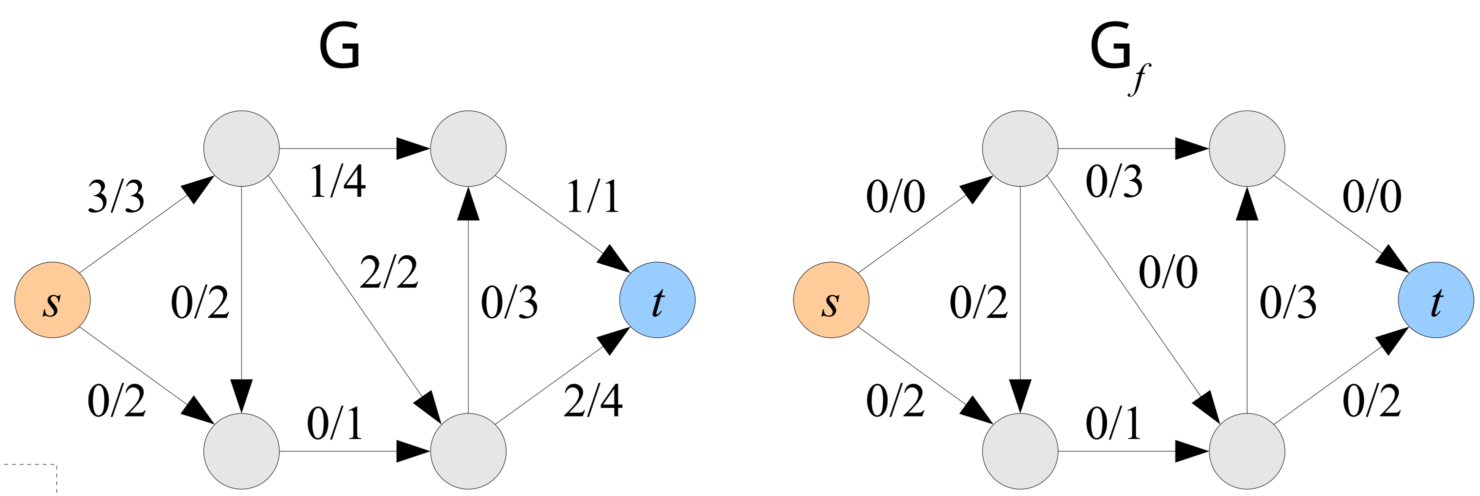
- Polynomial time
- Push-relabel methods
  - better performance for general graphs
  - e.g. Goldberg and Tarjan 1988: $O(V E \log(V^2/E))$
    - where V: number of vertices, E: number of edges
- Augmenting paths methods
  - iteratively push flow from source to sink along some path
  - better performance on specific graphs
  - e.g. Ford-Fulkerson 1956: $O(E \max|f|)$ for integer capacity c

To go further on this subject

Residual network/graph

- Given flow network $G = \langle V, E, c, f\rangle$

  Define residual network $G_f = \langle V, E, c_f, 0\rangle$ with
  - residual capacity $c_f(p,q) = c(p,q) - f(p,q)$
  - no flow, i.e., value 0 for all edges
- Example:



edge label: $f(p,q)/c(p,q)$

To go further on this subject

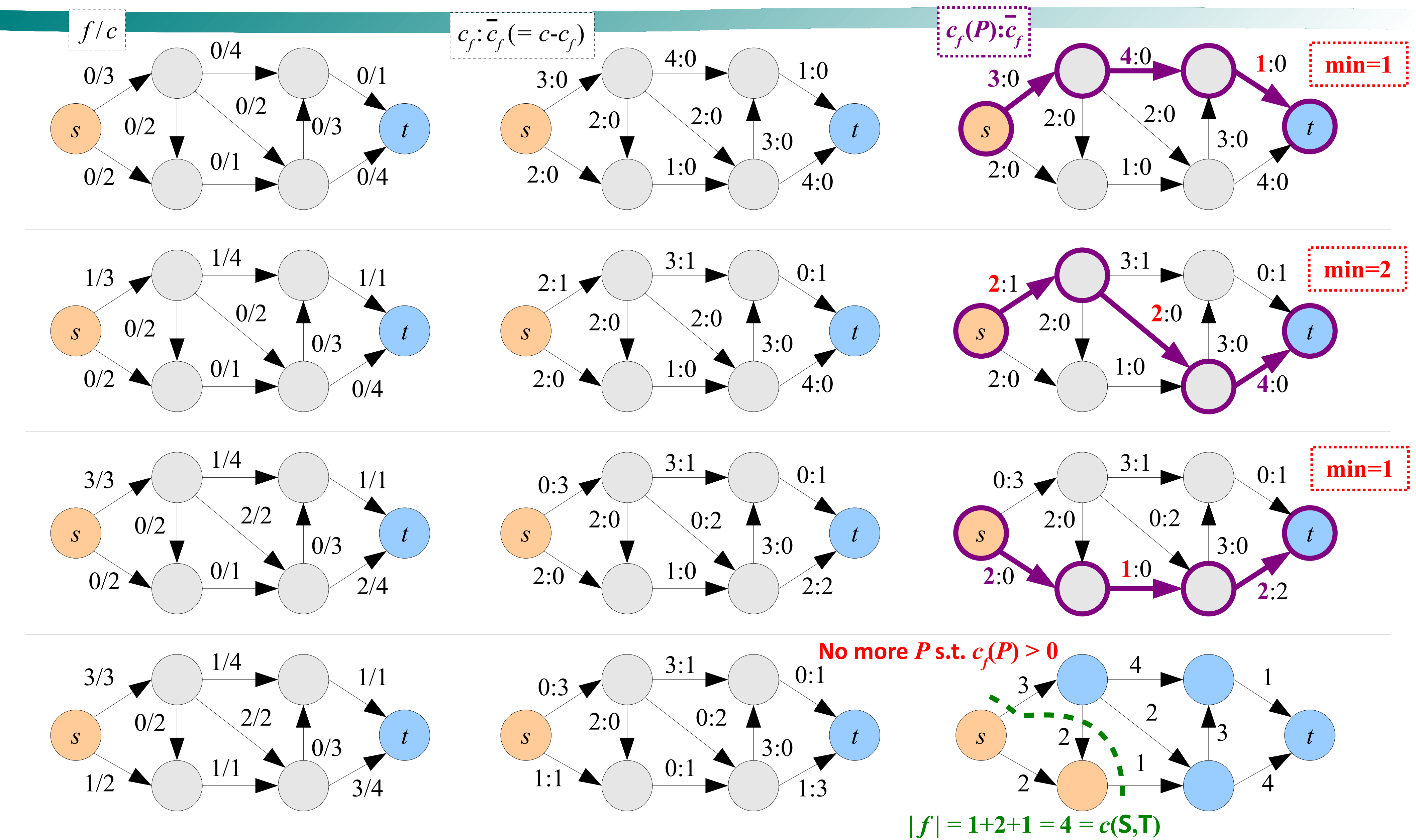
Ford-Fulkerson algorithm (1956)

termination = terminaison
semi-algorithm: termination not guaranteed for all inputs

$f(p,q) \leftarrow 0$ for all edges [P: augmenting path]
while $\exists$ path P from s to t such that $\forall(p,q)\in P\ \ c_f(p,q) > 0$
  $c_f(P) \leftarrow \min\{c_f(p,q) \mid (p,q) \in P\}$ [min residual capacity]
  for each edge $(p,q) \in P$
    $f(p,q) \leftarrow f(p,q) + c_f(P)$ [push flow along path]
    $f(q,p) \leftarrow f(q,p) - c_f(P)$ [keep skew symmetry]

- N.B. termination not guaranteed
  - maximum flow reached if (semi-)algorithm terminates (but may “converge” to less than maximum flow if it does not terminate)
  - always terminates for integer values (or rational values)

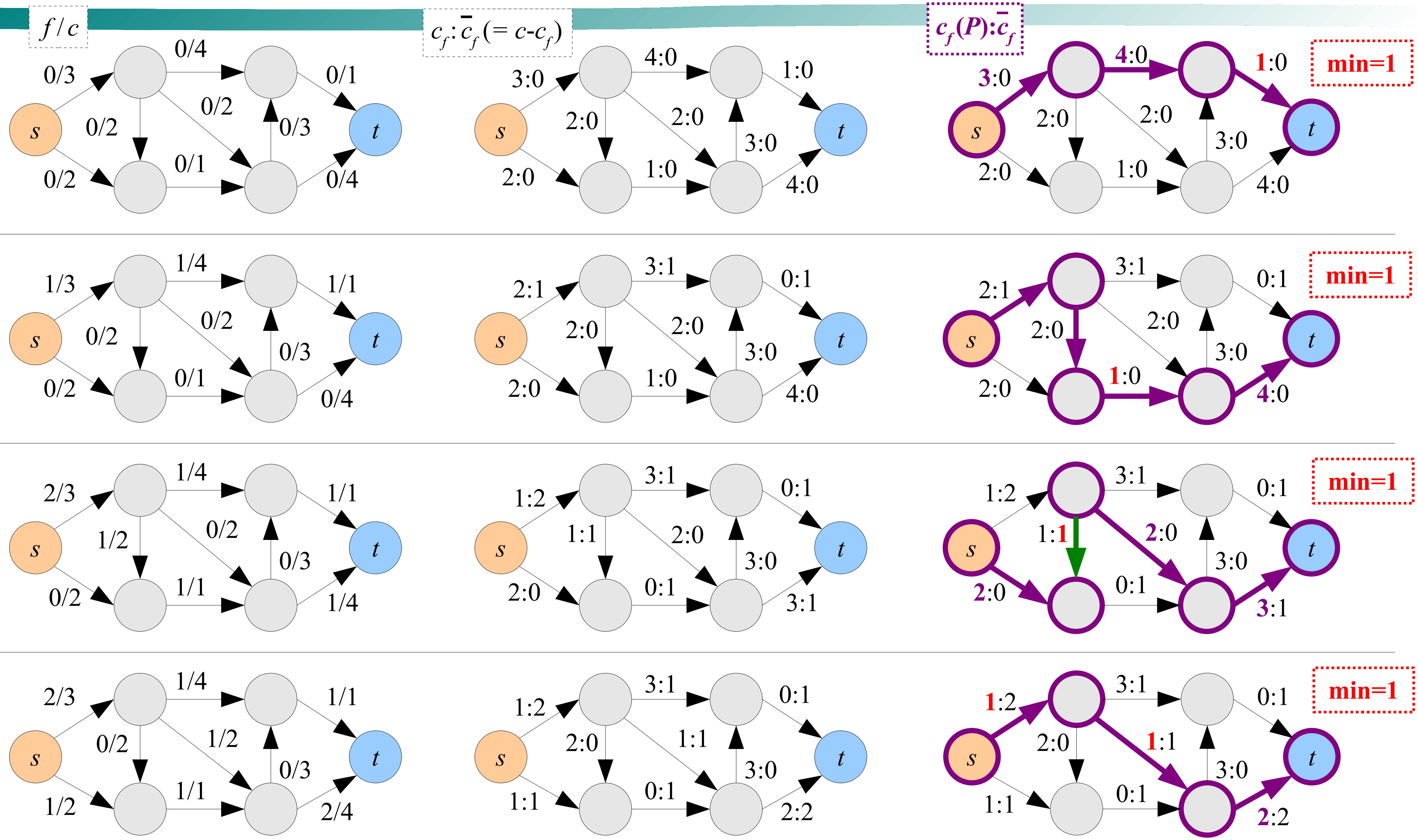
Ford-Fulkerson algorithm: an example



To go further on this subject

Ford-Fulkerson algorithm: an example

Taking edges backwards = OK (and sometimes needed)



$|f| = 1+1+1+1 = 4 = c(\mathbf{S},\mathbf{T})$

To go further on this subject

Edmonds-Karp algorithm (1972)

breadth-first = en largeur d'abord
sparse = épars, peu dense

- As Ford-Fulkerson but **shortest path** with >0 capacity
  - breadth-first search for augmenting path (cf. example above)
- Termination: now guaranteed
- Complexity: $O(V\,E^2)$
  - slower than push-relabel methods for general graphs
  - faster in practice for sparse graphs
- Other variant (Dinic 1970), complexity: $O(V^2\,E)$
  - other flow selection (blocking flows)
  - $O(V\,E \log V)$ with dynamic trees (Sleator & Tarjan 1981)

Maximum flow for grid graphs

- Fast augmenting path algorithm
  (Boykov & Kolmogorov 2004)
  - often significantly outperforms push-relabel methods
  - observed running time is linear
  - many variants since then
- But push-relabel algorithm can be run in parallel
  - good setting for GPU acceleration

The “best” algorithm depends on the context

To go further on this subject

Variant: Multiway cut problem

planar = planaire

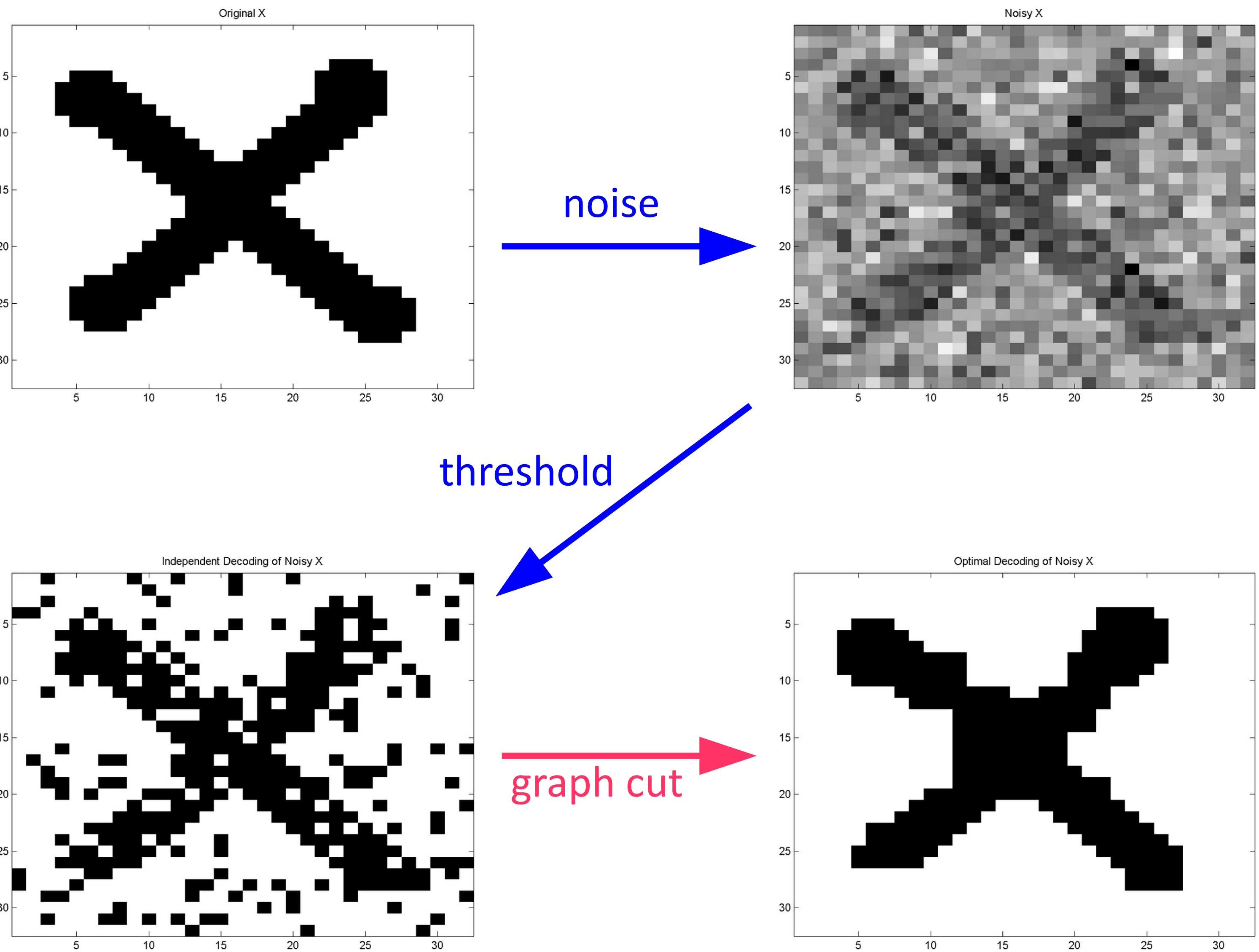
- More than two terminals: $\{s_1, \ldots, s_k\}$
- Multiway cut:
  - set of edges leaving each terminal in a separate component
- Multiway cut problem
  - find cut with minimum weight
  - same as min cut when $k = 2$
  - NP-hard if $k \geq 3$ (in fact APX-hard, i.e., NP-hard to approx.)
  - but can be solved exactly for planar graphs

Graph cuts for binary optimization

- **Inherently a binary technique**
  - **splitting in two**
- 1st use in image processing:
  binary image restoration (Greig et al. 1989)
  - black&white image with noise → image with no noise
- Can be generalized to large classes of binary energy
  - regular functions

Binary image restoration

noise = bruit
threshold = seuil

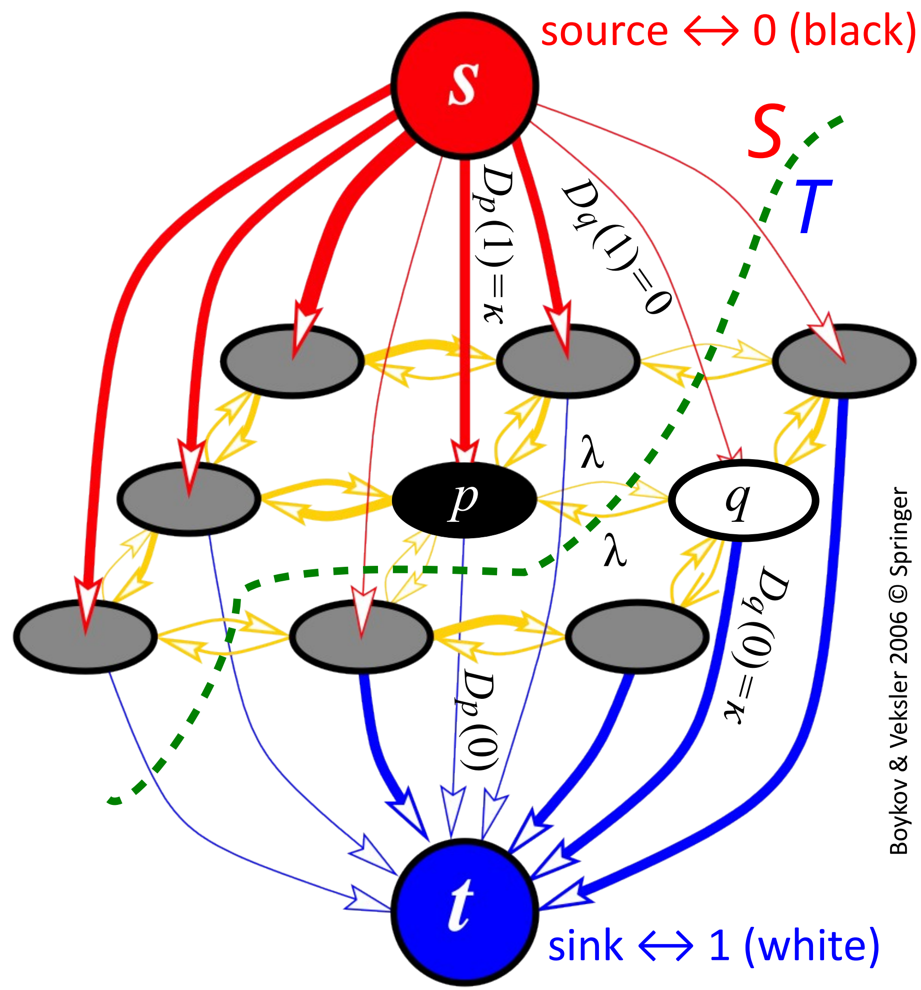


Binary image restoration: The graph cut view

penalty = pénalité, coût
reward = récompense

- Agreement with observed data
  - $D_p(l)$: penalty (= -reward) for assigning label $l \in \{0,1\}$ to pixel $p \in$ P
  - if $I_p = l$ then $D_p(l) < D_p(l')$ for $l' \neq l$
  - $w(s,p) = D_p(1)$, $w(p,t) = D_p(0)$
- Example:
  - if $I_p = 0$, $D_p(0) = 0$, $D_p(1) = \kappa$
    if $I_p = 1$, $D_p(0) = \kappa$, $D_p(1) = 0$
  - if $I_p = 0$ and $p \in$ S, $cost = D_p(0) = 0$
    if $I_p = 0$ and $p \in$ T, $cost = D_p(1) = \kappa$

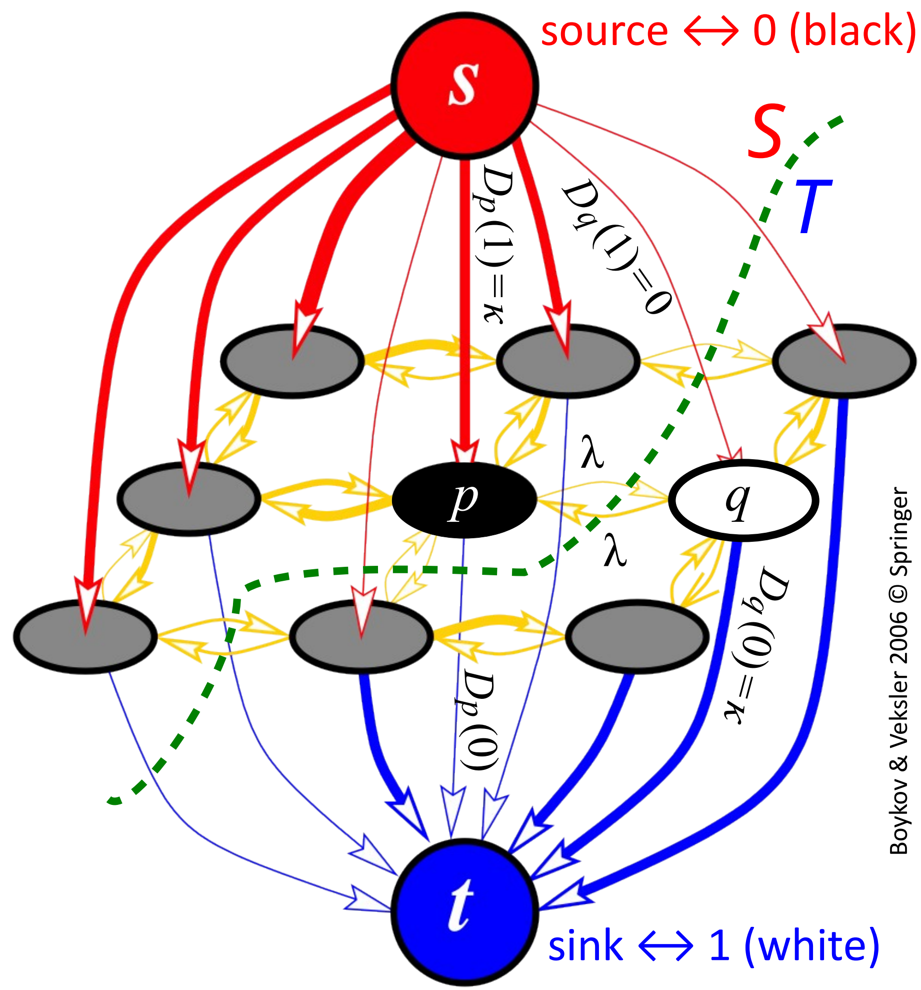
I_p : intensity of image I at pixel p



Binary image restoration: The graph cut view

penalty = pénalité, coût
reward = récompense
regularizing constraint = contrainte de régularisation
smoothing = lissage

- Agreement with observed data
  - $D_p(l)$: penalty (= −reward) for assigning label $l \in \{0,1\}$ to pixel $p \in$ P
  - if $I_p = l$ then $D_p(l) < D_p(l')$ for $l' \neq l$
  - $w(s,p) = D_p(1)$, $w(p,t) = D_p(0)$
- Minimize discontinuities
  - penalty for (long) contours
    - $w(p,q) = w(q,p) = \lambda > 0$
  - spatial coherence, regularizing constraint, smoothing factor... (see below)



Binary image restoration: The graph cut view

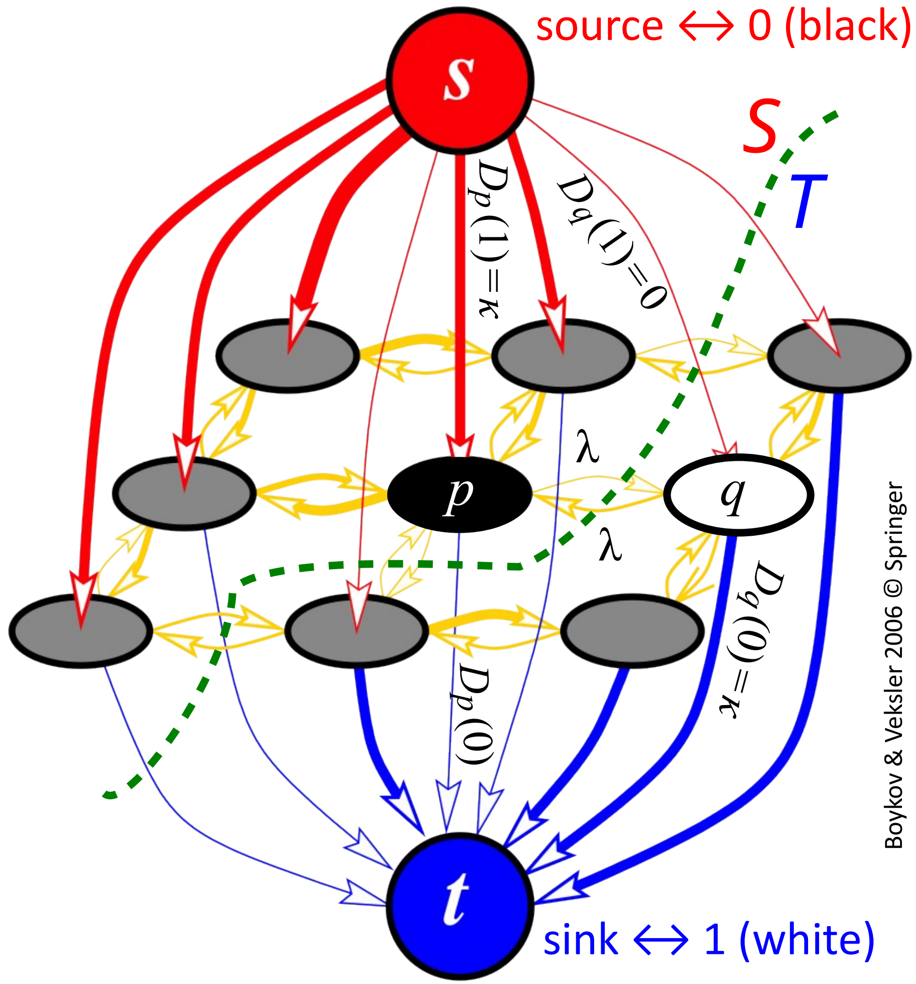
labeling = étiquetage

- Binary labeling f [N.B. different from "flow f"]
  - assigns label $f_p \in \{0,1\}$ to pixel $p \in$ P
    - $f : P \rightarrow \{0,1\} \quad f(p) = f_p$
- Cut $C = \{S,T\} \leftrightarrow$ labeling f
  - 1-to-1 correspondence: $f = \mathbf{1}_{|T}$
- Cost of a cut: $|C| =$

$$\Sigma_{p\in P} D_p(f_p) + \Sigma_{(p,q)\in S\times T}\, w(p,q)$$

= cost of flip + cost of local dissimilarity

- Restored image:
  = labeling corresponding to a minimum cut



Binary image restoration: The energy view

- Energy of labeling f
  - $E(f) \stackrel{\text{def}}{=} |C| =$

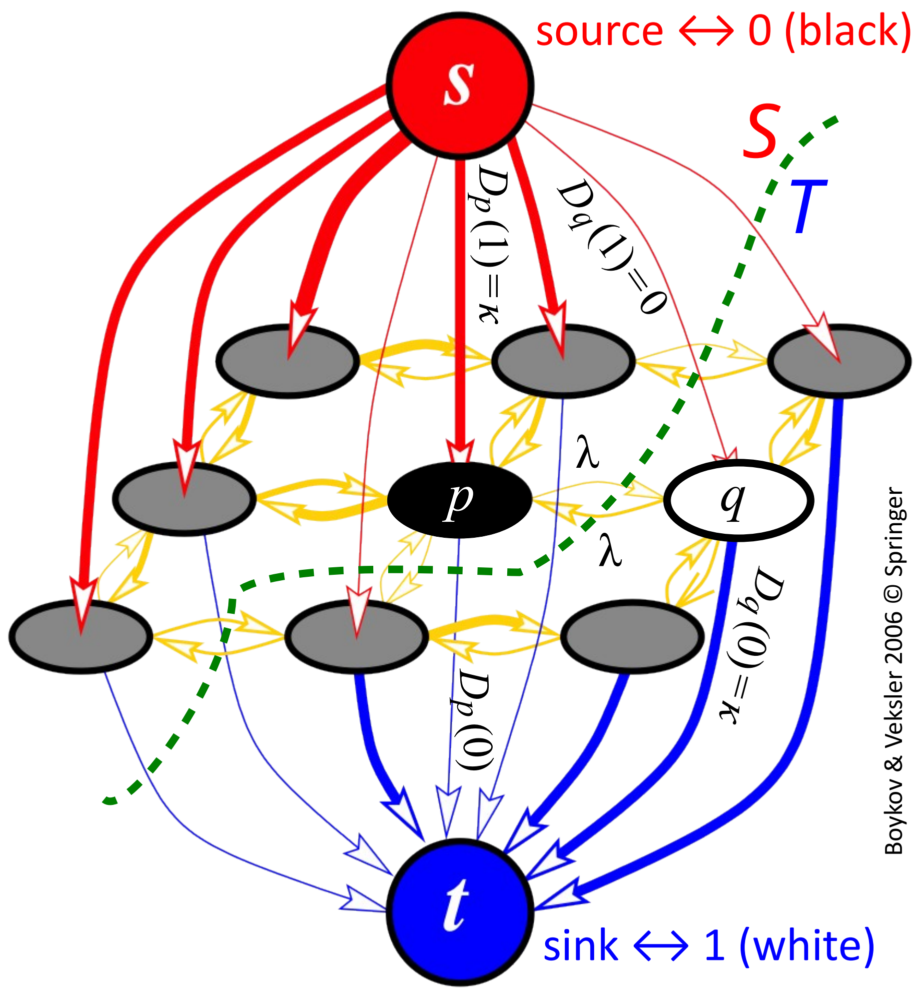
    $\Sigma_{p \in \mathrm{P}} D_p(f_p) +$

    $\lambda\, \Sigma_{(p,q) \in \mathrm{N}}\, \mathbf{1}(f_p = 0 \wedge f_q = 1)$

    where

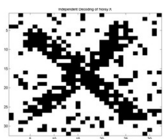
    $\mathbf{1}(\text{false}) = 0 \;\;|\;\; \mathbf{1}(\text{true}) = 1$

    [or: $\frac{1}{2}\lambda\, \Sigma_{(p,q) \in \mathrm{N}}\, \mathbf{1}(f_p \neq f_q)$]
- Restored image:
  - labeling corresponding to minimum energy (= minimum cut)

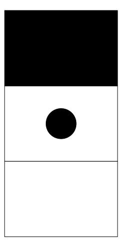


Binary image restoration: The smoothing factor

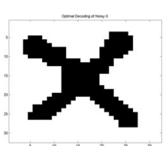
cluster = amas
outlier = point aberrant

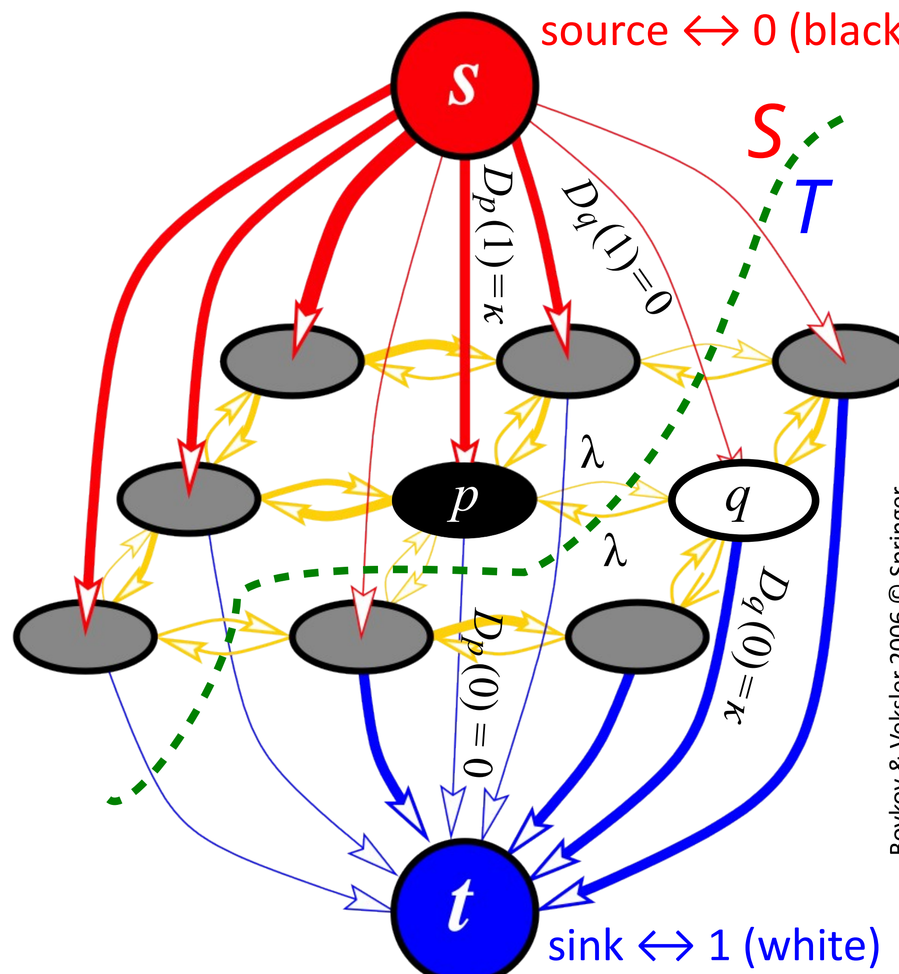


- Small λ (actually λ/κ):
  - pixels choose their label independently of their neighbors



- Large λ:
  - pixels choose the label with smaller average cost
- Balanced λ value:
  - pixels form compact, spatially coherent clusters with same label
  - noise/outliers conform to neighbors





Boykov & Veksler 2006 © Springer

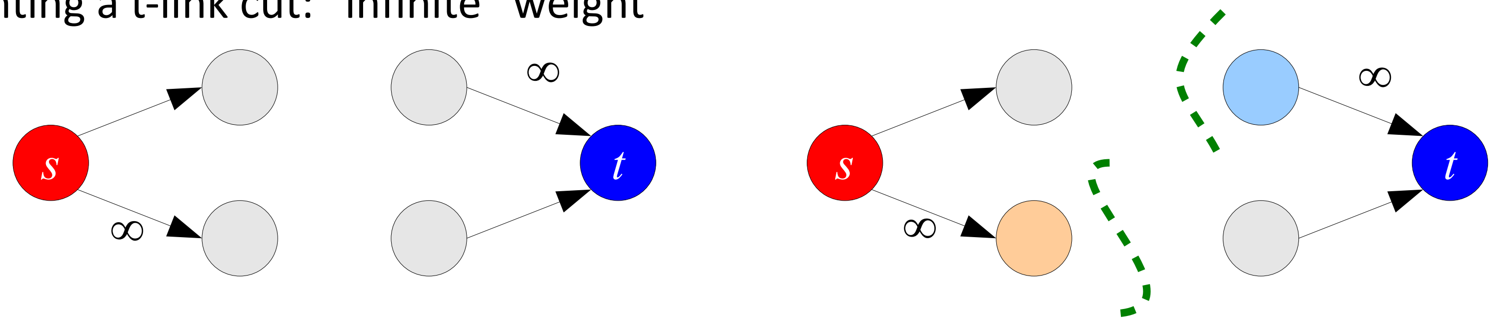
Graph cuts for energy minimization

- **Given some energy** $E(f)$ such that
  - $f : \mathsf{P} \to \mathsf{L} = \{0,1\}$ binary labeling
  - $E(f) = \underbrace{\sum_{p \in \mathsf{P}} D_p(f_p)}_{E_{\text{data}}(f)} + \underbrace{\sum_{(p,q) \in \mathsf{N}} V_{p,q}(f_p, f_q)}_{E_{\text{regul}}(f)}$
  - regularity condition (see below)
    - $V_{p,q}(0,0) + V_{p,q}(1,1) \le V_{p,q}(0,1) + V_{p,q}(1,0)$
- Theorem: **then there is a graph** whose minimum cut defines a labeling f that reaches the minimum energy (Kolmogorov & Zabih 2004)
  [N.B. Vladimir Kolmogorov, not Andrey Kolmogorov]

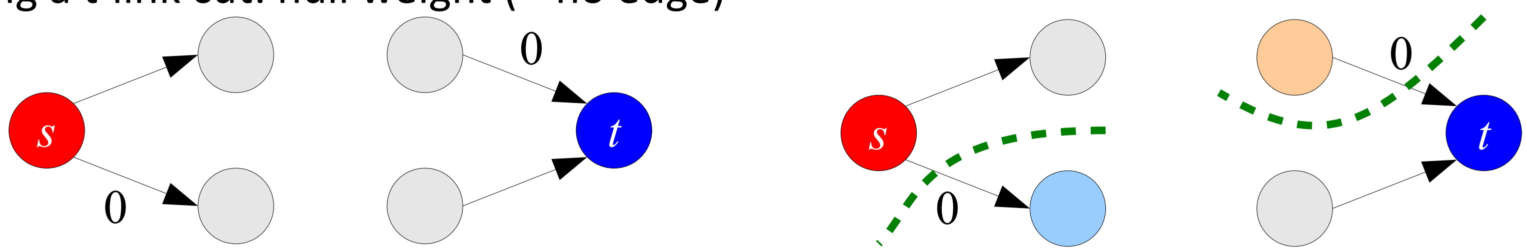
  [structure of graph somehow similar to above form]

Graph construction

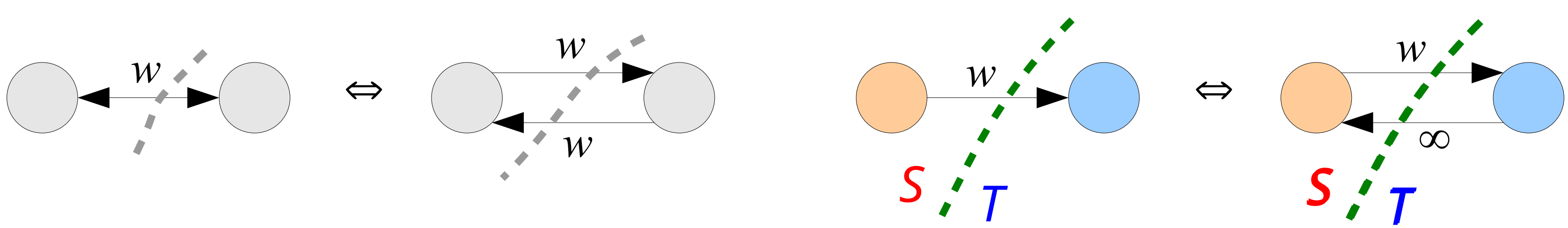
- Preventing a t-link cut: “infinite” weight



- Favoring a t-link cut: null weight (≈ no edge)



- Bidirectional edge vs monodirectional & back edges



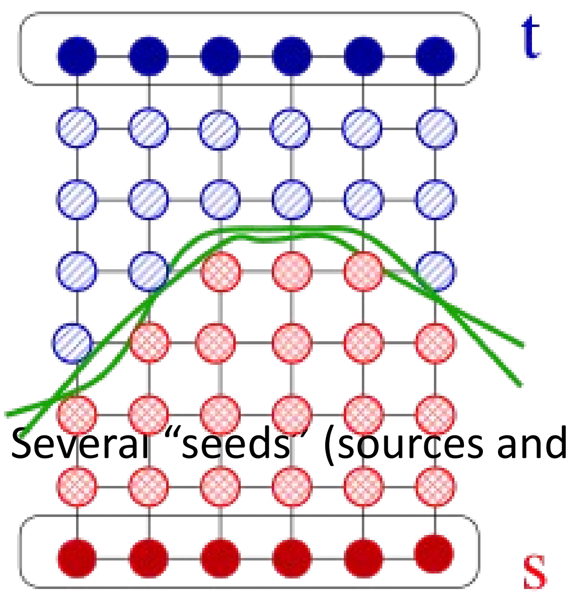
To go further on this subject

Graph cuts as hypersurfaces

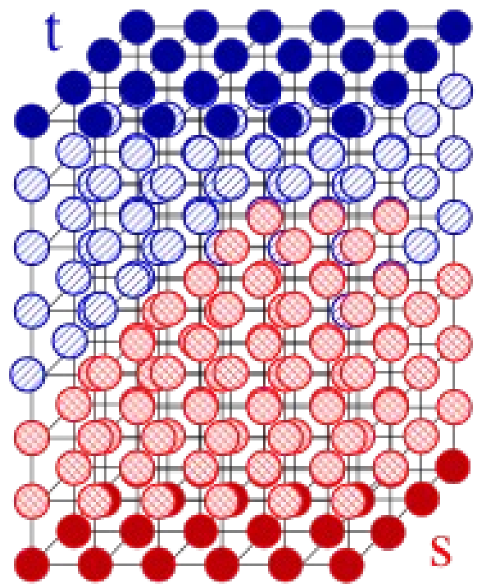
(cf. Boykov & Veksler 2006)

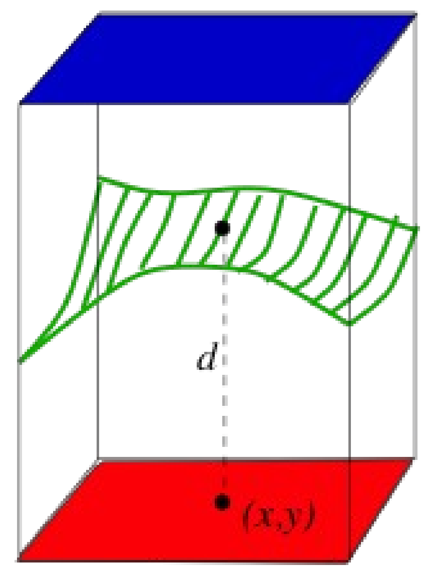
- Cut on a 2D grid
- Cut on a 3D grid

seed = graine



N.B. Several “seeds” (sources and sinks)





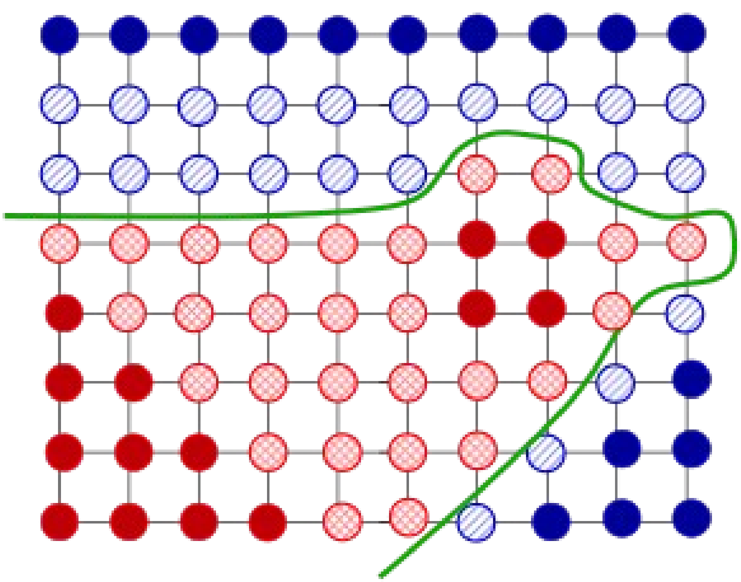
Boykov & Veksler 2006 © Springer

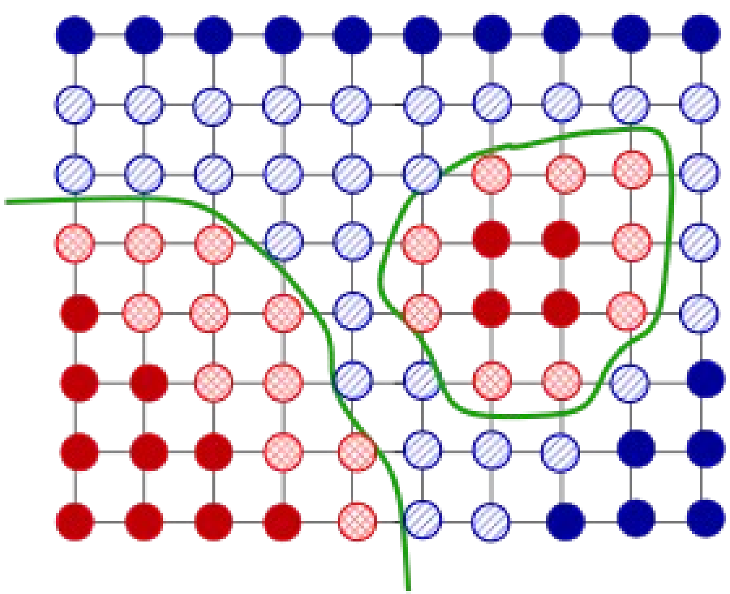
To go further on this subject

Example of topological issue

seed = graine

- Connected seeds
- Disconnected seeds



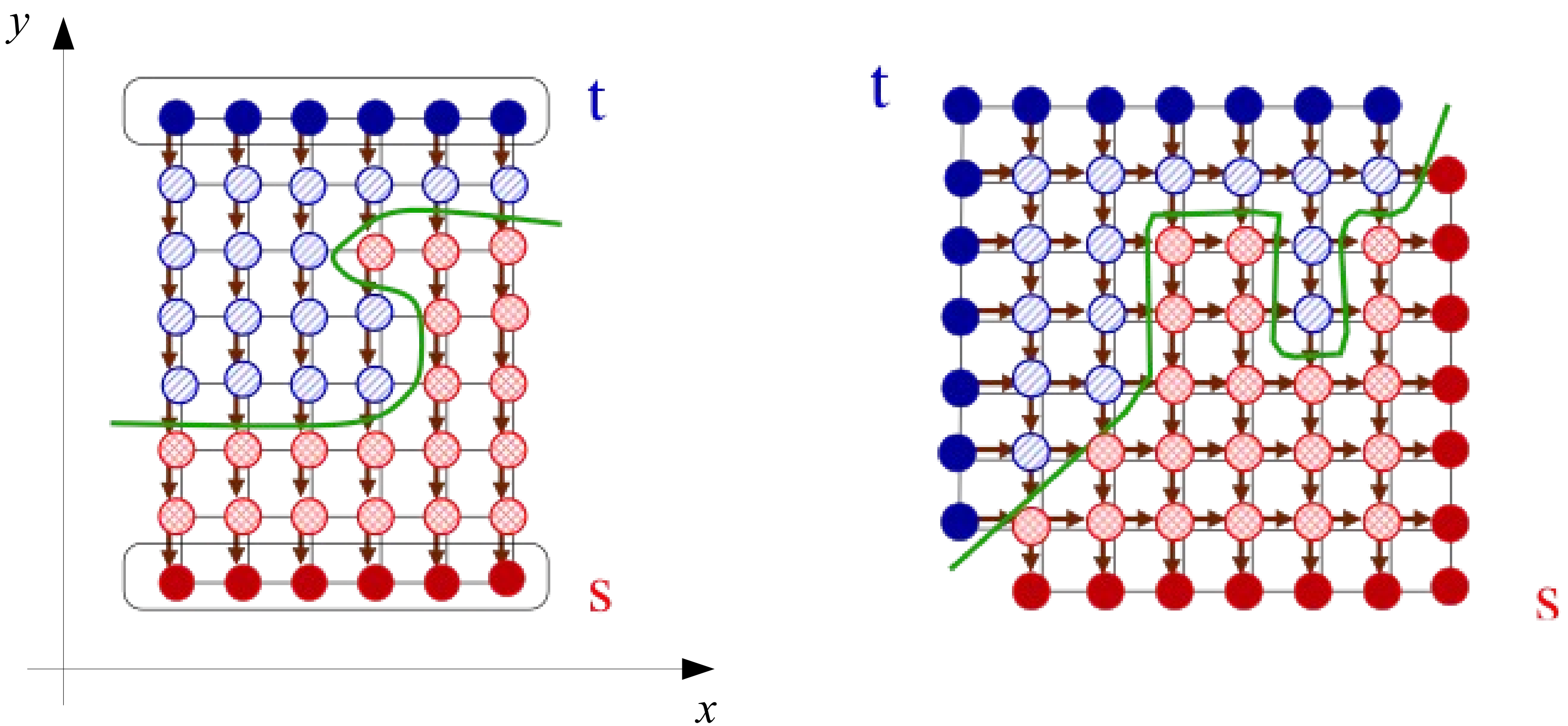


Boykov & Veksler 2006 © Springer

To go further on this subject

Example of topological constraint: fold prevention

- Ex. in disparity map estimation: $d = f(x,y)$
- In 2D: $y = f(x)$, only one value for y given one x



Boykov & Veksler 2006 © Springer

A “revolution” in optimization

simulated annealing = recuit simulé

- Previously (before Greig et al. 1989)
  - exact optimization like this was not possible
  - used approaches:
    - iterative algorithms such as simulated annealing
    - very far from global optimum, even in binary case like this
    - work of Greig et al. was (primarily) meant to show this fact
- Remained unnoticed for almost 10 years in the computer vision community...
    - maybe binary image restoration was viewed as too restrictive ?

      (Boykov & Veksler 2006)

Graph cut techniques: now very popular in computer vision

- Extensive work since 1998
  - Boykov, Geiger, Ishikawa, Kolmogorov, Veksler, Zabih and others...
- Almost linear in practice (in nb nodes/edges)
  - but beware of the graph size:
    it can be exponential in the size of the problem
- Many applications
  - regularization, smoothing, restoration
  - segmentation
  - stereovision: disparity map estimation, ...

Warning:
global optimum != best real-life solution

- Graph cuts provide exact, global optimum
  - to binary labeling problems (under regularity condition)
- But the problem remains a model
  - approximation of reality
  - limited number of factors
  - parameters (e.g., λ)

☛ Global optimum of **abstracted problem**,
not necessarily best solution **in real life**

Not for free

- Many papers construct
  - their own graph
  - for their own specific energy function
- The construction can be fairly complex

☛ Powerful tool but does not exempt from thinking
(contrary to some aspects of deep learning ☺)

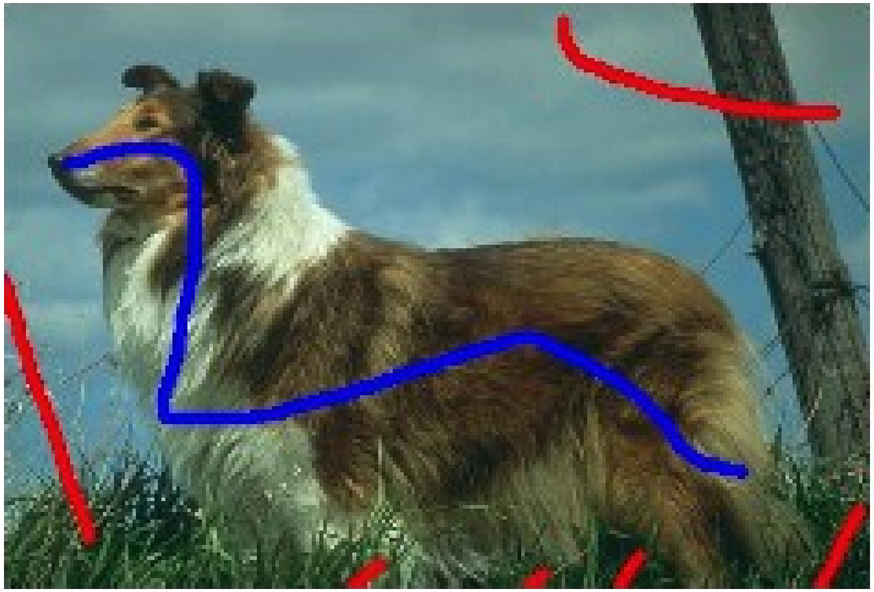
Graph cut vs deep learning

- Graph cut
  - works well, with proven optimality bounds
- Deep learning
  - works extremely well, but mainly empirical
- Somewhat complementary
  - graph cut sometimes used to regularize network output

Application to image segmentation

- Problem:
  - given an image with foreground objects and background
  - given sample areas of both kinds
  - separate objects from background

background = arrière-plan
sample = échantillon
area = zone

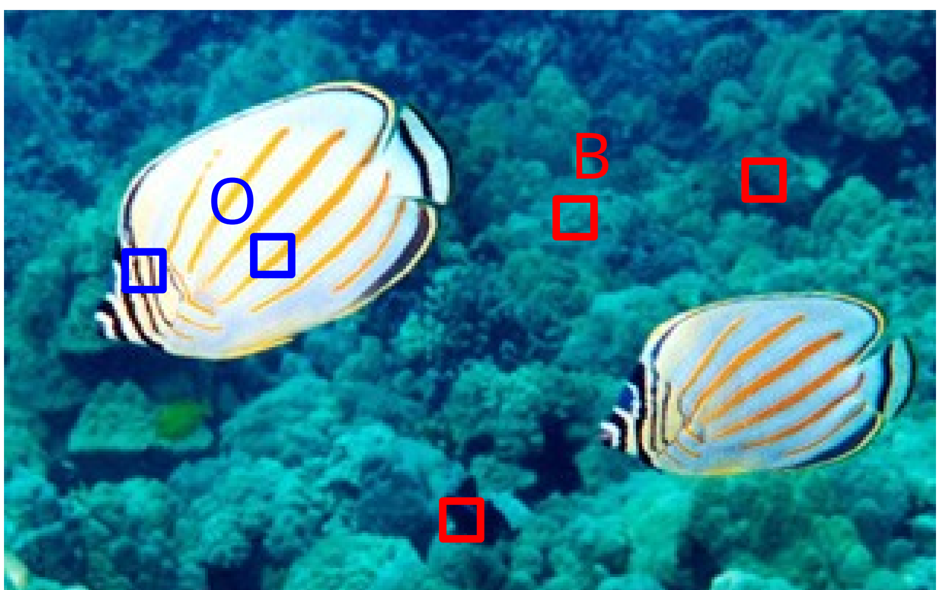


[Duchenne & al. 2008]

Application to image segmentation

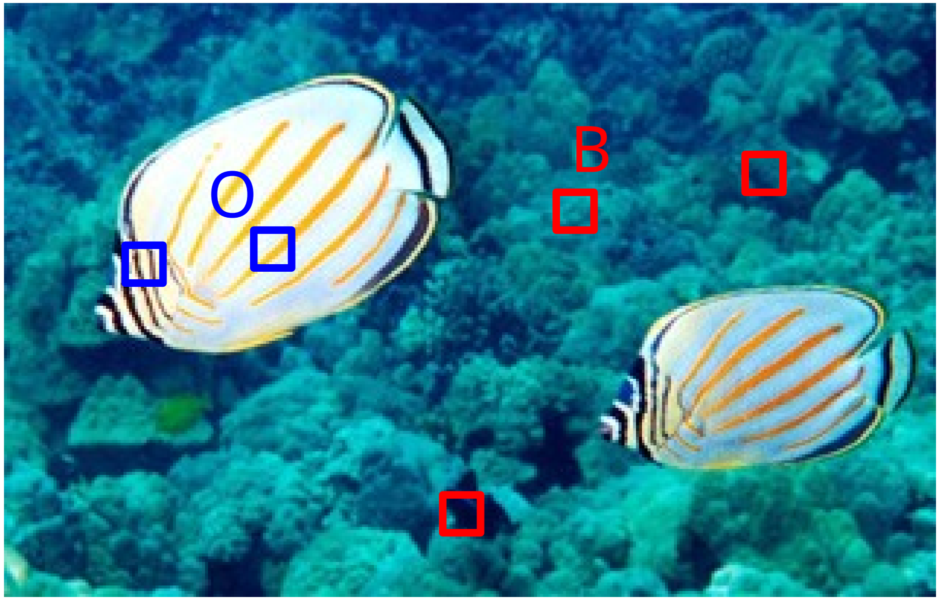
- Problem:
  - given an image with foreground objects and background
  - given sample areas of both kinds (O, B)
  - separate objects from background

background = arrière-plan
sample = échantillon
area = zone



Intuition

What characterizes an object/background segmentation ?

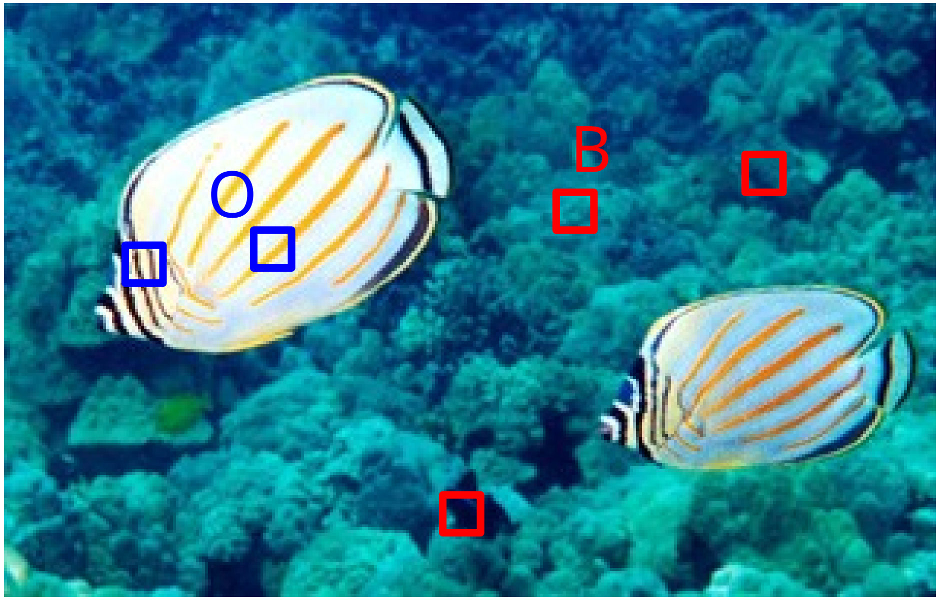


Intuition

background = arrière-plan
sample = échantillon
area = zone

What characterizes an object/background segmentation ?

- pixels of segmented object and background
  look like corresponding sample pixels O and B
- segment contours have high gradient, and are not too long



General formulation

[Boykov & Jolly 2001]

- Pixel labeling with binary decision $f_p \in L = \{0,1\}$
  - 1 = object, 0 = background
- Energy formulation
  - minimize $E(f) = D(f) + \lambda\, R(f)$
  - $D(f)$: **data term** (a.k.a. data fidelity term) = regional term
    - penalty for assigning labels f in image I given pixel sample assignments in L : O (object pixels), B (background pixels)
  - $R(f)$: **regularization term** = boundary term
    - penalty for label discontinuity of neighboring pixels
  - λ : relative importance of regularization term vs data term

data term = **terme d'attache aux données**
regularization term = **terme de régularisation**
a.k.a. = also known as
penalty = pénalité, coût
to assign = affecter (une valeur à qq chose)
sample = échantillon
background =
boundary = frontière
neighboring pixel = pixel voisin

To go further on this subject

Probabilistic justification/framework

posterior probability = probabilité a posteriori
likelihood = vraisemblance
(log-)likelihood = (log-)vraisemblance

- Minimize $E(f)$ ↔ maximize posterior proba. $\Pr(f\,|\,I)$
- Bayes theorem:

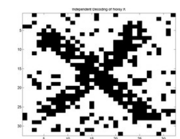
$$\boxed{\Pr(f\,|\,I)}\;\boxed{\Pr(I)} = \boxed{\Pr(I\,|\,f)}\;\boxed{\Pr(f)}$$

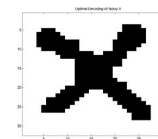
The term we want to maximize w.r.t. f

A constant (independent of f)

↔ data term, probability to observe image I knowing labeling f

↔ regularization term, depending on type of labeling and with various hypotheses (e.g., locality, cf. MRF below)





- Consider likelihoods $L(f\,|\,I) = \Pr(I\,|\,f)$
- Actually consider log-likelihoods (→ sums)

$$E(f) = D(f) + \lambda R(f) \;\Leftrightarrow\; -\log \Pr(f\,|\,I) + c = -\log \Pr(I\,|\,f) - \log \Pr(f)$$

To go further on this subject

Data term: linking estimated labels to observed pixels

penalty = pénalité, coût
to assign = affecter
sample = échantillon
likelihood = vraisemblance
random variable = variable aléatoire

- $D(f)$ and likelihood
  - penalty for assigning labels f in I given sample assignments ↔ (log-)likelihood that f is consistent with image samples
  - $D(f) = -\log L(f\,|\,I) = -\log \Pr(I\,|\,f)$
- Pixel independence hypothesis (common approximation)
  - $\Pr(I\,|\,f) = \prod_{p\in P} \Pr(I_p|\,f_p)$ if pixels iid
  - $D(f) = \sum_{p\in P} D_p(f_p)$ where $D_p(f_p) = -\log \Pr(I_p\,|\,f_p)$
    - $D_p(f_p)$: penalty for observing I_p for a pixel of type f_p

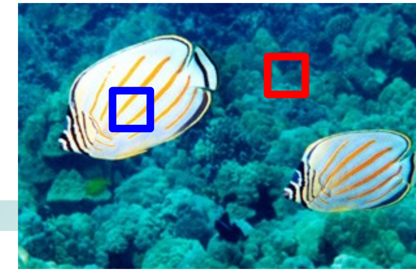
wrong strictly speaking, but "true enough" to be often assumed

☛ Find an estimate of $\Pr(I_p\,|\,f_p)$

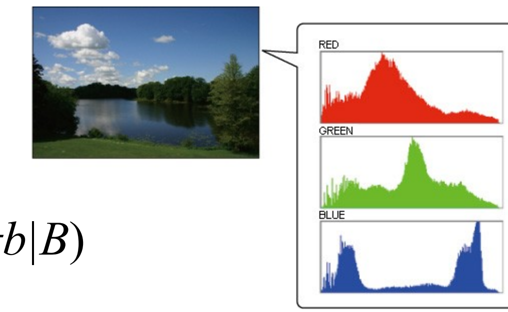
To go further on this subject

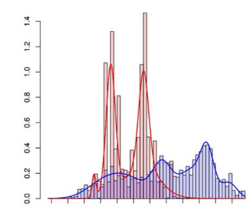
empirical probability =
probabilité empirique
(fréquence relative)
Gaussian mixture =
mélange de gaussiennes

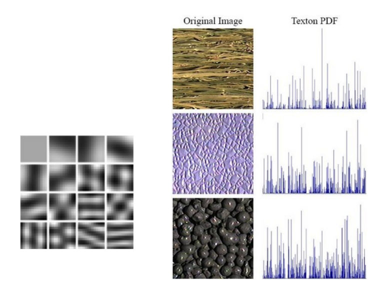
Data term: likelihood/color model



- Approaches to find an estimate of $\Pr(I_p|f_p)$
  - histograms
    - build an empirical distribution of the color of object/background pixels, based on pixels marked as object/background
    - estimate $\Pr(I_p\,|\,f_p)$ based on histograms: $\Pr_{\text{emp}}(rgb|O), \Pr_{\text{emp}}(rgb|B)$
  - Gaussian Mixture Model (GMM)
    - model the color of object (resp. background) pixels with a distribution defined as a mixture of Gaussians
  - texon (or texton): texture patch (possibly abstracted)
    - compare with expected texture property: response to filters (spectral analysis), moments...





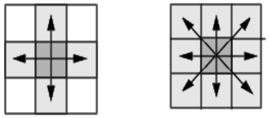


Blunsden 2006 @ U. of Edinburgh

To go further on this subject

Regularization term: locality hypotheses

- Markov random field (MRF), or Markov network
  - neighborhood system: $N = \{N_p \mid p \in P\}$
    - N_p : set neighbors of p such that $p \notin N_p$ and $p \in N_q \Leftrightarrow q \in N_p$



  - $X = (X_p)_{p \in P}$: field (set) of random variables such that each random variable X_p depends on other random variables only through its neighbors N_p

    ☛ **locality hypothesis:** $\Pr(X_p = x \mid X_{P\setminus\{p\}}) = \Pr(X_p = x \mid X_{N_p})$
  - $N \approx$ undirected graph: (p,q) edge iff $p \in N_q$ ($\Leftrightarrow q \in N_p$)

    (MRF also called undirected graphical model)

Markov random field = champ de Markov
random variable = variable aléatoire
neighborhood = voisinage
undirected graph = graph non orienté
graphical model = modèle graphique

To go further on this subject

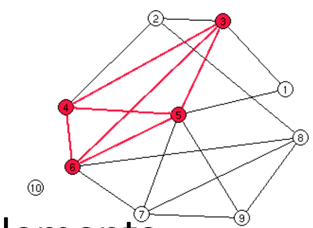
Regularization term: locality hypotheses

Gibbs random field = champ de Gibbs
undirected graph = graph non orienté
clique = clique (!)
clique potential = potentiel de clique
prior probability = probabilité a posteriori

- Gibbs random field (GRF)
  - *G* undirected graph, $X = (X_p)_{p \in P}$ random variables such that

$$\Pr(X = x) \propto \exp(-\textstyle\sum_{C \text{ clique of } G} V_C(x))$$

    - clique = complete subgraph: $\forall p \neq q \in C \;\; (p,q) \in G$
    - V_C: clique potential = prior probability of the given realization of the elements of the clique C (fully connected subgraph)



- Hammersley-Clifford theorem (1971)
  - If probability distribution has positive mass/density, i.e., if $\Pr(X = x) > 0$ for all x, then:

    X MRF w.r.t. graph *N* iff X GRF w.r.t. graph *N*

☛ provides a characterization of MRFs as GRFs

To go further on this subject

Regularization term: locality hypotheses

[Boykov, Veksler & Zabih 1998]

- Hypothesis 1: only 2nd-order cliques (i.e., edges)

$$R(f) = -\log \Pr(f) = -\log \exp(-\textstyle\sum_{(p,q)\text{ edge of } G} V_{(p,q)}(f)) \quad \text{[GRF]}$$
$$= \textstyle\sum_{(p,q)\in \mathrm{N}} V_{p,q}(f_p, f_q) \quad \text{[MRF \textbf{pairwise potentials}]}$$

- Hypothesis 2: (generalized) Potts model

$$V_{p,q}(f_p,f_q) = B_{p,q}\ \mathbf{1}(f_p \neq f_q)$$

i.e.,
$$V_{p,q}(f_p,f_q) = 0 \qquad \text{if } f_p = f_q$$
$$V_{p,q}(f_p,f_q) = B_{p,q} \qquad \text{if } f_p \neq f_q$$

(Origin: statistical mechanics
- spin interaction in crystalline lattice
- link with "energy" terminology)

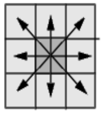
pairwise = par paire
pairwise potential = potentiel d'ordre 2
Potts model = modèle de Potts
statistical mechanics = physique statistique

To go further on this subject

Examples of boundary penalties (ad hoc)

- Penalize label discontinuity at intensity continuity
  - $B_{p,q} = \exp(-(I_p - I_q)^2/2\sigma^2) \,/\, \mathrm{dist}(p,q)$

    [Boykov & Jolly 2001]
    - large between pixels of similar intensities, i.e., when $|\, I_p - I_q \,| < \sigma$
    - small between pixels of dissimilar intensities, i.e., when $|\, I_p - I_q \,| > \sigma$
    - decrease with pixel distance $\mathrm{dist}(p,q)$ [here: 1 or $\sqrt{2}$]
    - ≈ distribution of noise among neighboring pixels
- Penalize label discontinuity at low gradient
  - $B_{p,q} = g(\|\, \nabla I_p \,\|)$ with g positive decreasing

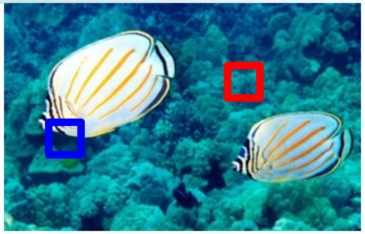
    
    - e.g., $g(x) = 1/(1 + c\, x^2)$
    - penalization for label discontinuity at low gradient

To go further on this subject

Wrapping up

- Pixel labeling with binary decision $f_p \in \{0,1\}$
  - 0 = background, 1 = object
- Energy formulation
  - minimize $E(f) = D(f) + \lambda\, R(f)$
  - data term: $D(f) = \sum_{p \in \mathrm{P}} D_p(f_p)$
    - $D_p(f_p)$: penalty for assigning label f_p to pixel p given its color/texture
  - regularization term: $R(f) = \sum_{(p,q) \in \mathrm{N}} B_{p,q}\, \mathbf{1}(f_p \neq f_q)$
    - $B_{p,q}$: penalty for label discontinuity between neighbor pixels p, q
  - λ : relative importance of regularization term vs data term

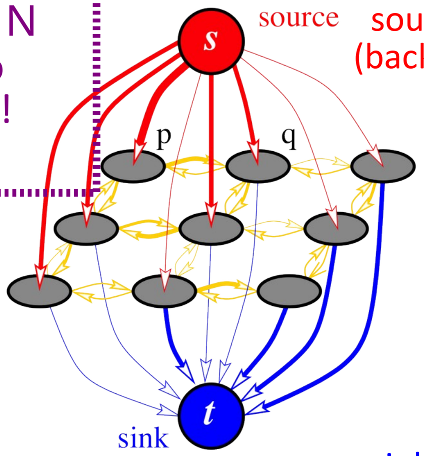
Graph-cut formulation (version 1)



- Direct expression as graph-cut problem:
  - $V = \{s,t\} \cup P$
  - $E = \{(s,p) \mid p \in P\} \cup \{(p,q) \mid p,q \in N\} \cup \{(p,t) \mid p \in P\}$

Warning:
the connexity N
is specific to
the problem!

Edge	Weight	Sites
(p,q)	$\lambda\, B_{p,q}$	$(p,q) \in N$
(s,p)	$D_p(1)$	$p \in P$
(p,t)	$D_p(0)$	$p \in P$

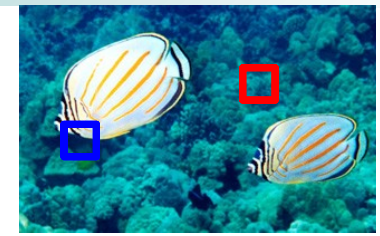


source ↔ 0
(background)

sink ↔ 1
(foreground object)

  - $E(f) = \sum_{p \in P} D_p(f_p) + \lambda \sum_{(p,q) \in N} B_{p,q}\, \mathbf{1}(f_p \neq f_q)$
    - ex. $D_p(l) = -\log \Pr_{emp}(I_p \mid f_p = l)$ [empirical probability for O et B]
    - ex. $B_{p,q} = \exp(-(I_p - I_q)^2 / 2\sigma^2) \,/\, \mathrm{dist}(p,q)$

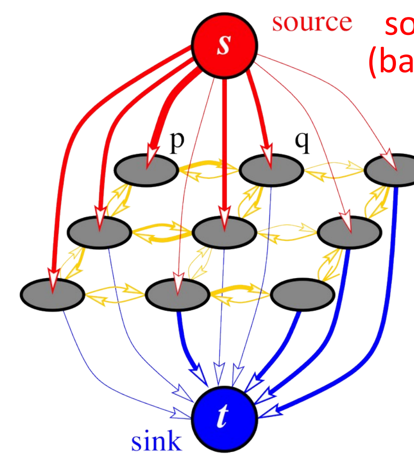
Graph-cut formulation (version 1)



- Direct expression as graph-cut problem:
  - $\mathrm{V} = \{s,t\} \cup \mathrm{P}$
  - $E = \{(s,p) \mid p \in \mathrm{P}\} \cup \{(p,q) \mid p,q \in \mathrm{N}\} \cup \{(p,t) \mid p \in P\}$
  -

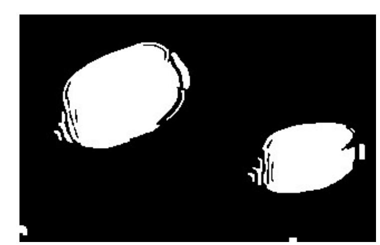
Edge	Weight	Sites
(p,q)	$\lambda\, B_{p,q}$	$(p,q) \in \mathrm{N}$
(s,p)	$D_p(1)$	$p \in \mathrm{P}$
(p,t)	$D_p(0)$	$p \in \mathrm{P}$

  - $E(f) = \sum_{p \in \mathrm{P}} D_p(f_p) + \lambda \sum_{(p,q) \in \mathrm{N}} B_{p,q}\, \mathbf{1}(f_p \neq f_q)$
- Any problem/risk with this formulation ?

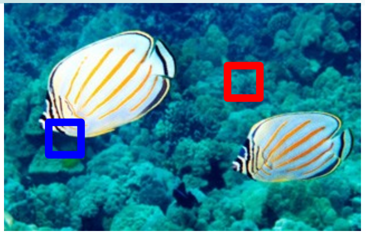


source ↔ 0
(background)

sink ↔ 1
(foreground object)

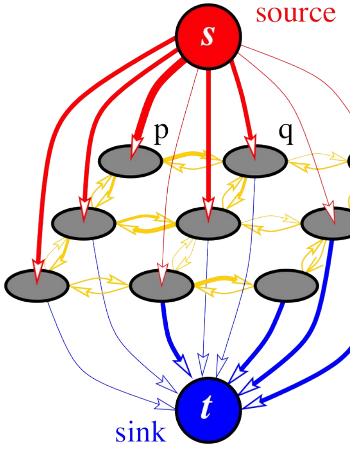


Graph-cut formulation (version 1)



- Direct expression as graph-cut problem:
  - $\mathrm{V} = \{s,t\} \cup \mathrm{P}$
  - $E = \{(s,p) \mid p \in \mathrm{P}\} \cup \{(p,q) \mid p,q \in \mathrm{N}\} \cup \{(p,t) \mid p \in P\}$
  -

Edge	Weight	Sites
(p,q)	$\lambda\, B_{p,q}$	$(p,q) \in \mathrm{N}$
(s,p)	$D_p(1)$	$p \in \mathrm{P}$
(p,t)	$D_p(0)$	$p \in \mathrm{P}$

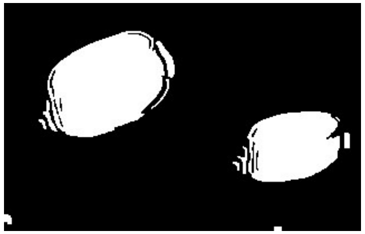


source ↔ 0
(background)

sink ↔ 1
(foreground object)

  - $E(f) = \sum_{p \in \mathrm{P}} D_p(f_p) + \lambda \sum_{(p,q) \in \mathrm{N}} B_{p,q}\, \mathbf{1}(f_p \neq f_q)$

- Pb: pixels of object/background samples not <u>necessarily</u> assigned with good label !

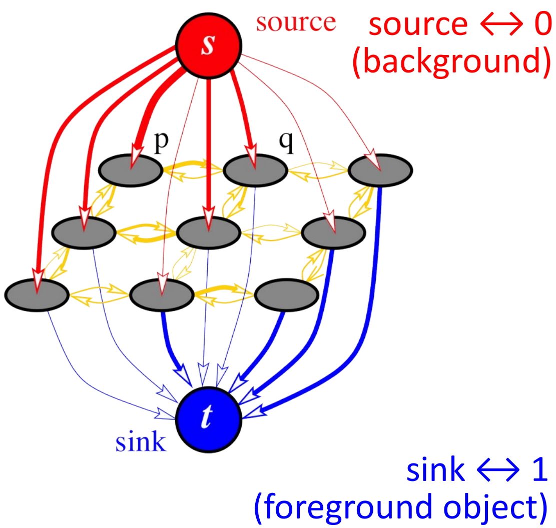


Graph-cut formulation (version 2)

[Boykov & Jolly 2001]

- Obj/Bg samples now always labeled OK in minimal f^*

Edge	Weight	Sites
(p,q)	$\lambda\, B_{p,q}$	$(p,q) \in \mathrm{N}$
(s,p)	$D_p(1)$	$p \in \mathrm{P},\ p \notin (\mathrm{O} \cup \mathrm{B})$
	K	$p \in \mathrm{B}$
	0	$p \in \mathrm{O}$
(p,t)	$D_p(0)$	$p \in \mathrm{P},\ p \notin (\mathrm{O} \cup \mathrm{B})$
	0	$p \in \mathrm{B}$
	K	$p \in \mathrm{O}$

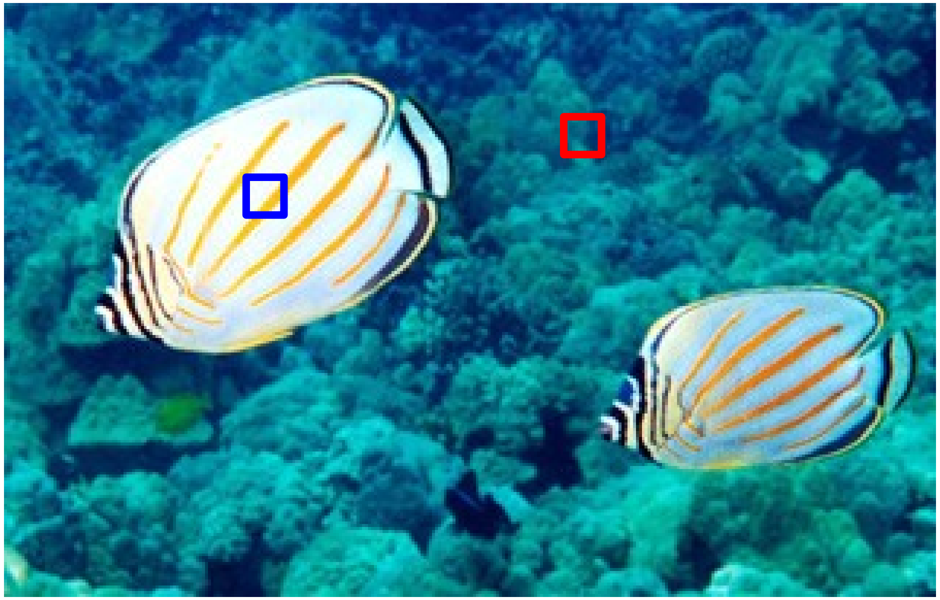


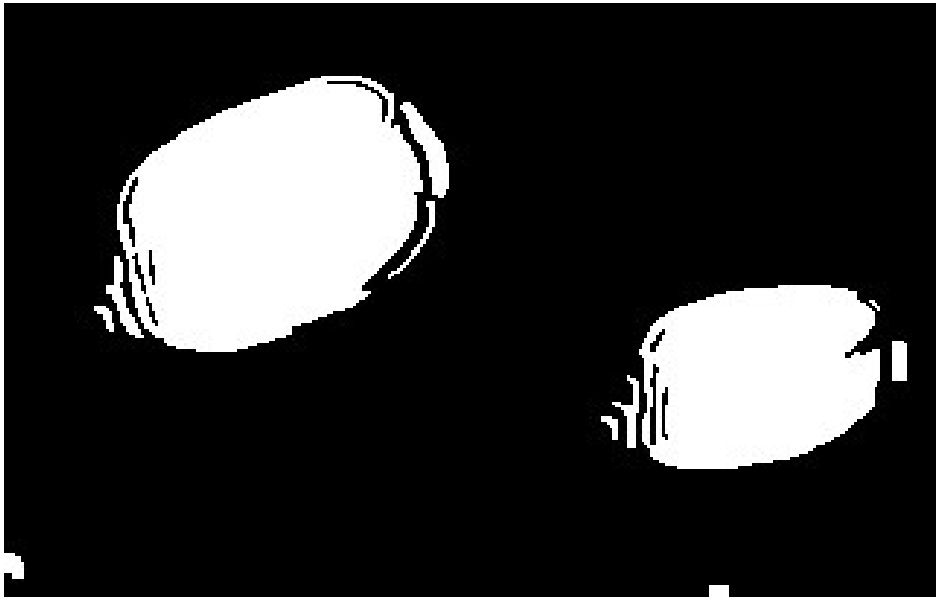
- where $K = 1 + \max_{p\in\mathrm{P}} \lambda \sum_{(p,q)\in\mathrm{N}} B_{p,q}$
  $K \approx +\infty$, i.e., too expensive to pay $\Rightarrow$ label never assigned

To go further on this subject

Some limitations
(here with simple color model)

- Is the segmentation OK ?





To go further on this subject

Some limitations (here with simple color model)

color model not complex enough

sensitivity to regularization parameter

neighboring model not complex enough

Part 2

Multi-label problems

Exact vs approximate solutions

Application to stereovision
(disparity/depth map estimation):
disparity/depth $\leftrightarrow$ label

Two-label (binary) problem

- P : set of sites (pixels, voxels...)
- N : set of neighboring site pairs
- L $= \{0,1\}$: binary labels
- f : P $\rightarrow$ L binary labeling [notation: $f_p = f(p) = l$]
- $E : (P \rightarrow L) \rightarrow \mathbb{R}$: energy
  - $$E(f) = \underbrace{\textstyle\sum_{p\in \mathrm{P}} D_p(f_p)}_{E_{\mathrm{data}}(f)} + \underbrace{\textstyle\sum_{(p,q)\in \mathrm{N}} V_{p,q}(f_p, f_q)}_{E_{\mathrm{regul}}(f)}$$
    - $D_p(l)$: label penalty for site p
    - $V_{p,q}(l,l')$: prior knowledge about optimal pairwise labeling
- Pb: find f^* that reaches the minimum energy $E(f^*)$

Two-label problem assumptions

- $E(f) = \sum_{p \in \mathrm{P}} D_p(f_p) + \sum_{(p,q) \in \mathrm{N}} V_{p,q}(f_p, f_q)$
- $D_p(l)$: label penalty for site p
  - small/null for preferred label, large for undesired label
  - assumption $D_p(l) \geq 0$ (else add constant → same optimum)
- $V_{p,q}(l,l')$: prior knowledge on optimal pairwise labeling
  - in general, smoothness: non-decreasing function of $\mathbf{1}(l \neq l')$
    - e.g., $V_{p,q}(l,l') = u_{p,q}\ \mathbf{1}(l \neq l')$ [Potts model]
- Regularity condition, required for min-cut ($\Rightarrow c(p,q) \geq 0$)
  - $V_{p,q}(0,0) + V_{p,q}(1,1) \leq V_{p,q}(0,1) + V_{p,q}(1,0)$ [see below]

Multi-label problem

disparity = disparité

- P : set of sites (pixels, voxels...)
- N : set of neighboring site pairs
- L : finite set of labels (→ can model scalar or even vector)
  - e.g., **discretization** of intensity, stereo disparity, motion vector...
- f : P → L labeling
- $E : (P \to L) \to \mathbb{R}$: energy
  - $E(f) = \sum_{p \in \mathrm{P}} D_p(f_p) + \sum_{(p,q) \in \mathrm{N}} V_{p,q}(f_p, f_q) = E_{\mathrm{data}}(f) + E_{\mathrm{regul}}(f)$
    - $D_p(l)$: label penalty for site p
    - $V_{p,q}(l_p, l_q)$: prior knowledge about optimal pairwise labeling

- Pb: find f^* that reaches the minimum energy $E(f^*)$

Multi-label problem assumptions

smoothness = lissage

- $E(f) = \sum_{p \in \mathrm{P}} D_p(f_p) + \sum_{(p,q) \in \mathrm{N}} V_{p,q}(f_p\ , f_q)$
- $D_p(l)$: label penalty for site p
  - small for preferred label, large for undesired label
  - assumption $D_p(l) \geq 0$ (else add constant → same optimum)
- $V_{p,q}(l_p,l_q)$: prior knowledge on optimal pairwise labeling
  - in general, smoothness prior:
    non-decreasing function of $\| l_p - l_q \|$ [norm used if vector]
    - e.g., $V_{p,q}(l_p,l_q) = \lambda_{p,q} \| l_p - l_q \|$
    - smaller penalty for closer labels

Graph cuts for "general" energy minimization

- Problem: find labeling $f^*: \mathsf{P} \to \mathsf{L}$ minimizing energy

$$E(f) = \sum_{p\in\mathsf{P}} D_p(f_p) + \sum_{(p,q)\in\mathsf{N}} V_{p,q}(f_p, f_q)$$

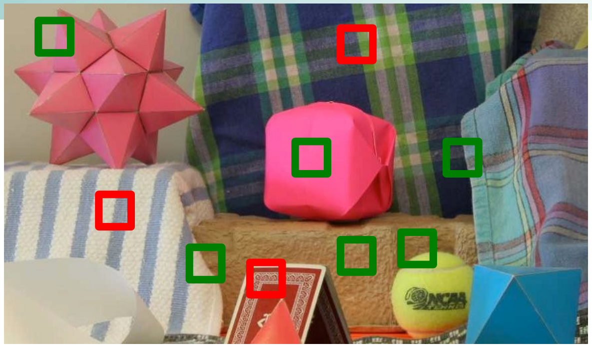
- Question: can a **globally optimal** labeling f^* be found using some graph-cut construction?
- Answer:
  - binary labeling: yes <u>iff</u> $V_{p,q}$ is regular (Kolmogorov & Zabih 2004)

    $V_{p,q}(0,0) + V_{p,q}(1,1) \leq V_{p,q}(0,1) + V_{p,q}(1,0)$ [otherwise NP-hard]
  - multi-labeling: yes if $V_{p,q}$ convex (Ishikawa 2003)

    and if L linearly ordered (⇒ 1D only ⇒ not 2D motion vector)
  - otherwise: approximate solutions (but some very good)

Piecewise-smooth vs everywhere-smooth

piecewise = par morceaux



- Observation: object properties often smooth everywhere except on boundaries

☛ Consequence: <u>piecewise-smooth</u> models more appropriate than <u>everywhere-smooth</u> models

original

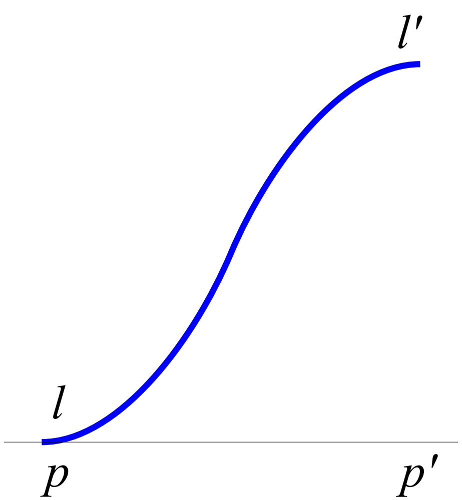
uniform smoothing

piecewise smoothing

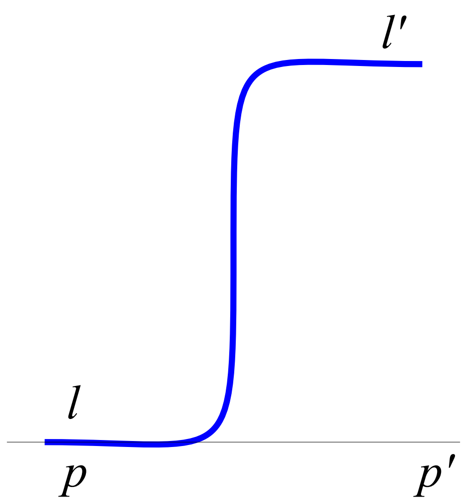
Piecewise-smooth models vs everywhere-smooth models

steep = raide, très pentu

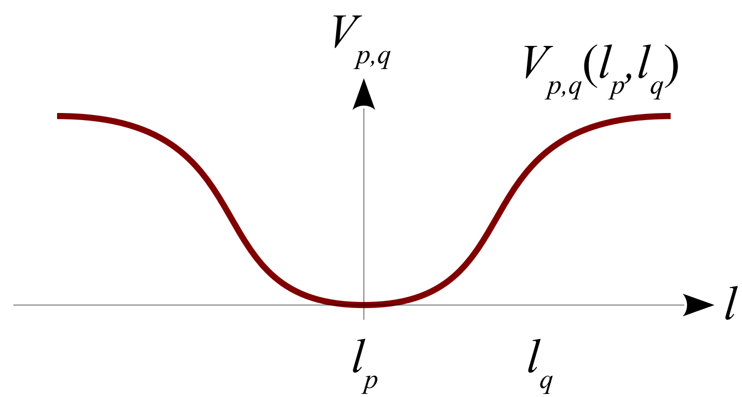
- Local variation of potentials $V_{p,q}$ depending on label variation



locally smooth from l to l' when going from p to p'



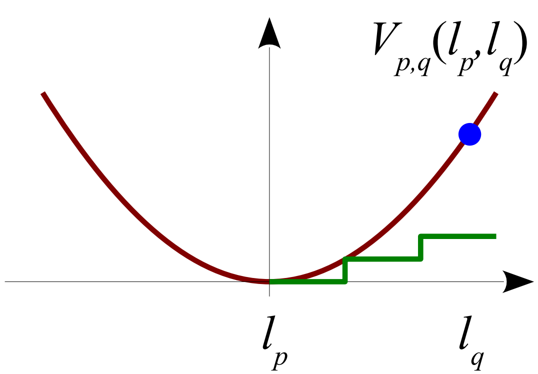
locally steep from l to l' when going from p to p'



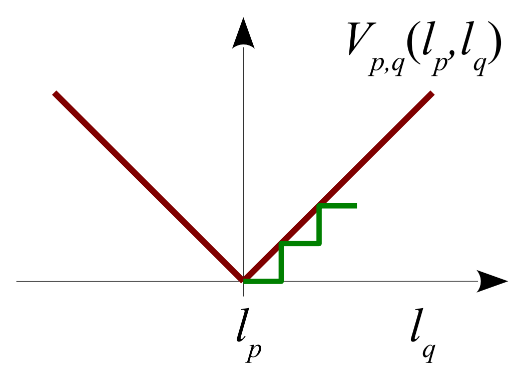
piecewise-smooth potential

Piecewise-smooth potentials vs everywhere-smooth potentials

- General graph construction for any convex $V_{p,q}$ (Ishikawa 2003)
  - convex ⇒ large penalty for sharp jump
  - a few small jumps cheaper than one large jump
  - ☛ discontinuities smoothed with “ramp” ⇒ oversmoothing

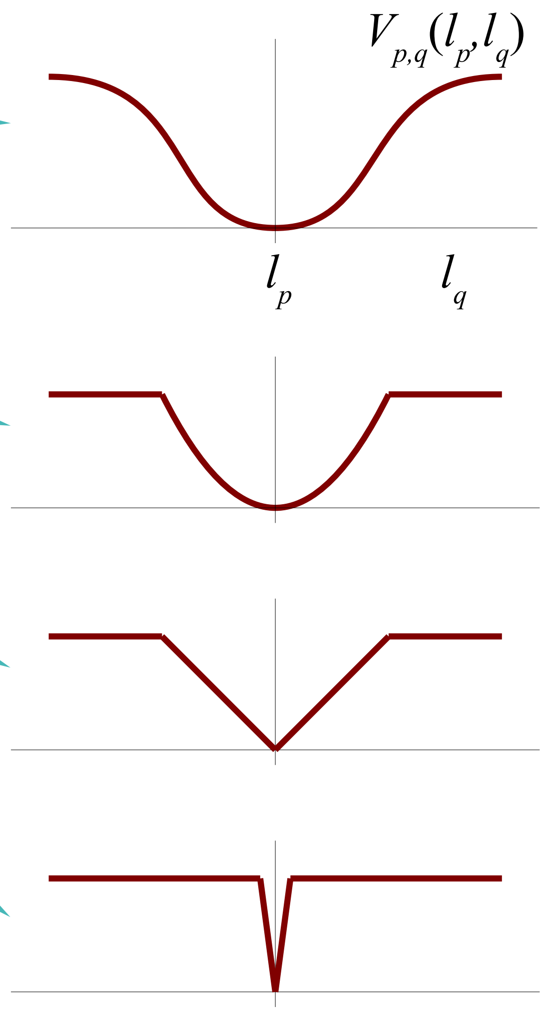


In practice, best results with “least convex” function, e.g.,
$V_{p,q}(l_p,l_q) = \lambda_{p,q} \, || \, l_p - l_q \, ||$



Discontinuity-preserving energy

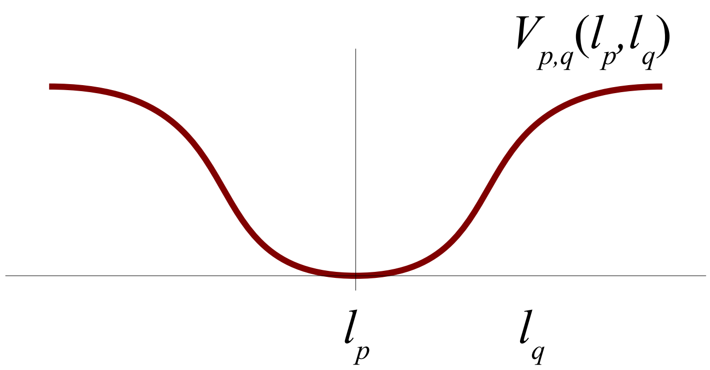
- At edges, very different labels for adjacent pixels are OK
- To not overpenalize in E adjacent but very different labels:
  - $V_{p,q}$ non-convex function of $\| l_p - l_q \|$
  - for instance (cap max):
    - $V_{p,q} = \min(K, \| l_p - l_q \|^2)$
    - $V_{p,q} = \min(K, \| l_p - l_q \|)$
    - $V_{p,q} = u_{p,q}\, \mathbf{1}(l_p \neq l_q)$ (Potts model)



Difficulty of minimization

simulated annealing = recuit simulé

- $E(f) = \sum_{p \in \mathrm{P}} D_p(f_p) + \sum_{(p,q) \in \mathrm{N}} V_{p,q}(f_p, f_q)$ with
  - $f : \mathrm{P} \to \mathrm{L}$
  - $V_{p,q}(f_p, f_q)$ non convex
- $\min_f E(f)$: minimization of non-convex function in large-dimension space (dimension = $|\mathrm{P}|$)
  - NP-hard even in simple cases
    - e.g. $V_{pq}(f_p, f_q) = \mathbf{1}(f_p \neq f_q)$ (Potts model) with $|\mathrm{L}| > 2$
  - general case: simulated annealing...



Exact binary optimization (reminder)

- Pb: find labeling $f^*: \mathsf{P} \rightarrow \mathsf{L} = \{0,1\}$ minimizing energy

  $$E(f) = \sum_{p \in \mathsf{P}} D_p(f_p) + \sum_{(p,q) \in \mathsf{N}} V_{p,q}(f_p, f_q)$$
- Question:
  - can a **globally optimal** labeling f^* be found using some graph-cut construction?
- Answer (Kolmogorov & Zabih 2004):
  - yes iff V_{pq} is regular
    - $V_{p,q}(0,0) + V_{p,q}(1,1) \leq V_{p,q}(0,1) + V_{p,q}(1,0)$
  - otherwise it's NP-hard
- But what about **general energies** on **binary** variables ?

Exact binary optimization

[Kolmogorov & Zabih 2004]

- Question:
  - what functions can be minimized using graph cuts?
- Classes of functions on binary variables:
  - F^2: $E(x_1,\ldots,x_n) = \sum_i E^i(x_i) + \sum_{i<j} E^{i,j}(x_i,x_j)$
  - F^3: $E(x_1,\ldots,x_n) = \sum_i E^i(x_i) + \sum_{i<j} E^{i,j}(x_i,x_j) + \sum_{i<j<k} E^{i,j,k}(x_i,x_j,x_k)$
  - F^m: $E(x_1,\ldots,x_n) = \sum_i E^i(x_i) + \ldots + \sum_{u_1<\ldots<u_m} E^{u_1,\ldots,u_m}(x_{u_1},\ldots,x_{u_m})$

***m*-th order** potentials

- "Using graph cuts": E graph-representable iff

  $\exists$ graph $\mathrm{G} = \langle \mathrm{V},\mathrm{E} \rangle$ with $\mathrm{V} = \{v_1,\ldots,v_n, s, t\}$ such that

  $\forall$configuration $\mathbf{x}=x_1,\ldots,x_n$,

  $E(x_1,\ldots,x_n) = \mathrm{cost}($min s-t-cut in which $v_i \in S$ if $x_i=0$ and $v_i \in T$ if $x_i=1) + k$ constant $\in \mathbb{R}$

Exact binary optimization

[Kolmogorov & Zabih 2004]

- E regular iff
  - F^2: $\forall i,j \;\; E^{i,j}(0,0) + E^{i,j}(1,1) \leq E^{i,j}(0,1) + E^{i,j}(1,0)$
  - F^m: for all terms $E^{u_1,\ldots,u_m}$ in E, all projections (specializations) of $E^{u_1,\ldots,u_m}$ to a two-variable function (i.e., all variables fixed but two) are regular
- Question:
  - what functions can be minimized using graph cuts?
- Answer (Kolmogorov & Zabih 2004):
  - F^2, F^3: E graph-representable $\Leftrightarrow$ E regular
  - any binary E: E not regular $\Rightarrow$ E not graph-representable

To go further on this subject

Link with submodularity

submodular = sous-modulaire

- $g : 2^{\mathrm{P}} \to \mathbb{R}$ submodular
  - iff $g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y)$

    for any $X, Y \subset \mathrm{P}$
  - iff $g(X \cup \{j\}) - g(X) \geq g(X \cup \{i,j\}) - g(X \cup \{i\})$

    for any $X \subset \mathrm{P}$ and $i,j \in \mathrm{P} \setminus X$
- g submodular $\Leftrightarrow$ E regular, with $E(\mathbf{x}) = \mathrm{g}(\{\, p \in \mathrm{P} \mid x_p{=}1 \})$
  - $E^{i,j}(0,1) + E^{i,j}(1,0) \geq E^{i,j}(0,0) + E^{i,j}(1,1)$
- $\exists$ independent results on submodular functions
  - minimization in polynomial time but slow, best known $O(n^6)$

Exact multi-label optimization (for 2nd-order potentials)

- Problem: find labeling $f^*: \mathsf{P} \to \mathsf{L}$ minimizing energy

  $E(f) = \sum_{p \in \mathsf{P}} D_p(f_p) + \sum_{(p,q) \in \mathsf{N}} V_{p,q}(f_p, f_q)$
- Assumption: L linearly ordered — w.l.o.g. $\mathsf{L} = \{1,\ldots,k\}$

  (1D only ⇒ not suited, e.g., for 2D motion vector estimation)
- Solution: reduction/encoding to binary label case
  - for $V_{p,q}(l_p,l_q) = \lambda_{p,q}\,|\,l_p - l_q\,|$ (Boykov et al. 1998, Ishikawa & Geiger 1998)
  - for any convex $V_{p,q}$ (Ishikawa 2003)
- See also
  - MinSum pbs (Schlesinger & Flach 2006)
  - submodular $V_{p,q}$ (Darbon 2009)

linearly ordered = linéairement ordonné
w.l.o.g. = without loss of generality

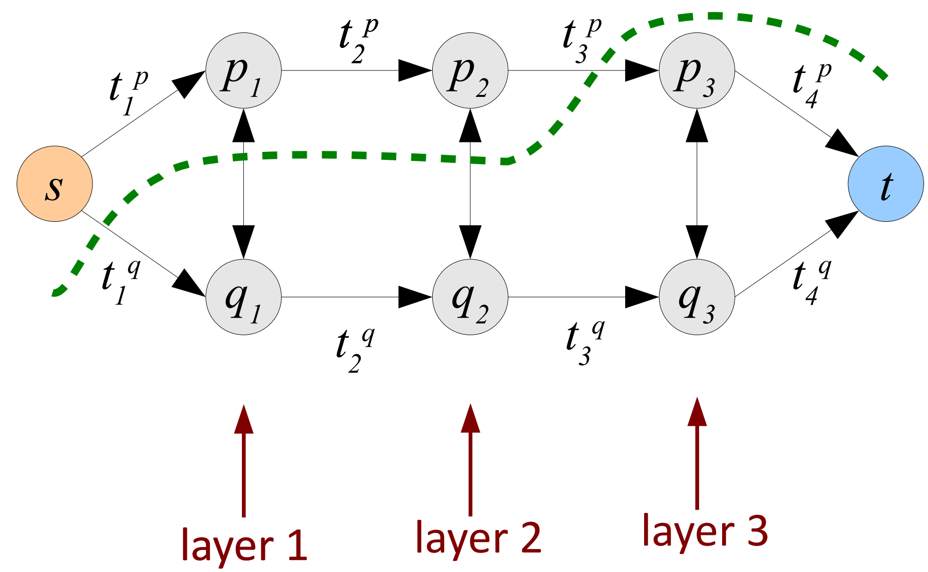
Linear multi-label graph construction

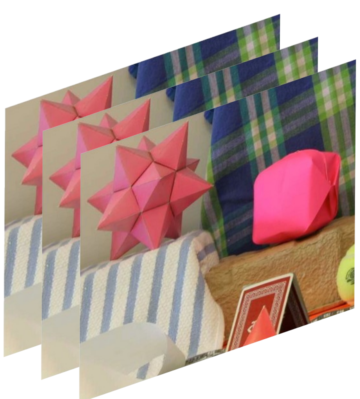
(cf. Boykov et al. 1998)

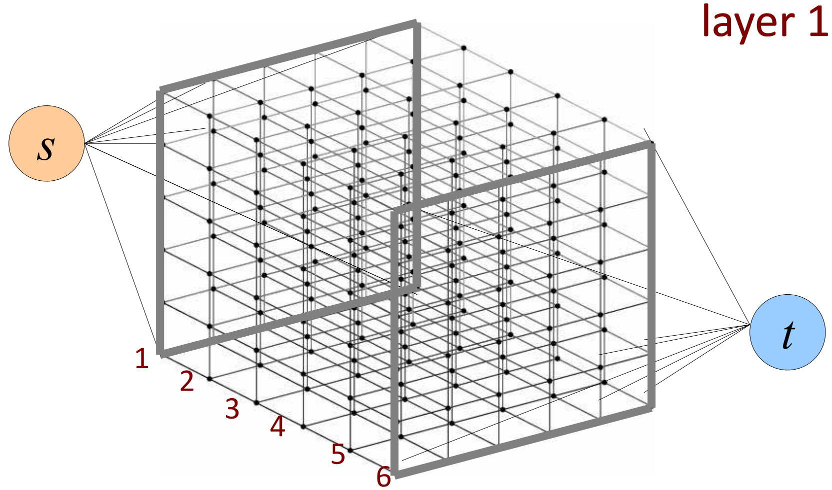
- Given L $= \{1,\ldots,k\}$
- General idea:
  - construct one layer per label value
  - read label value from cut location

e.g., $k = 4$

cut: $f_p=3, f_q=1$







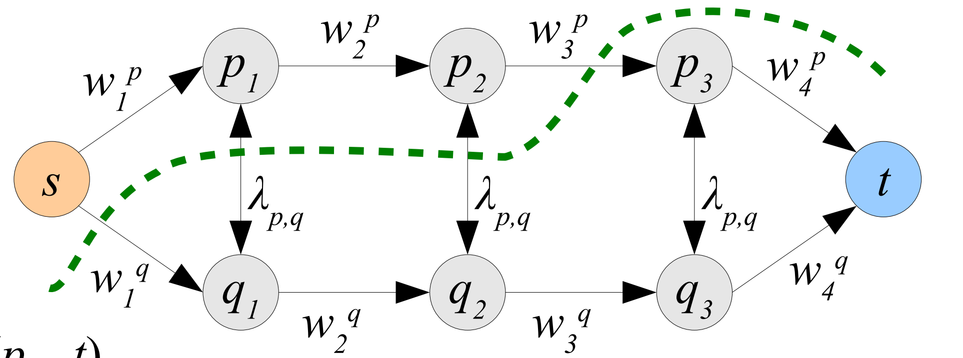
layer = couche

Linear multi-label graph construction

(cf. Boykov et al. 1998)

- $E(f) = \sum_{p\in P} D_p(f_p) + \sum_{(p,q)\in N} \lambda_{p,q} |f_p - f_q|$
  with $f_p \in L = \{1,\ldots,k\}$

cut: $f_p = 3, f_q = 1$



Attempt 1:

- For each site p
  - create nodes $p_1,\ldots,p_{k-1}$
  - create edges $t_1^p = (s,p_1)$, $t_j^p = (p_{j-1},p_j)$, $t_k^p = (p_{k-1},t)$
  - assign weights $w_j^p = w(t_j^p) = D_p(j)$
- For each pair of neighboring sites p and q
  - create edges $(p_j,q_j)_{j\in\{1,\ldots,k-1\}}$ with weight $\lambda_{p,q}$
- Read label value from cut location, e.g., $p_2 \in S$, $p_3 \in T \Rightarrow f_p = 3$

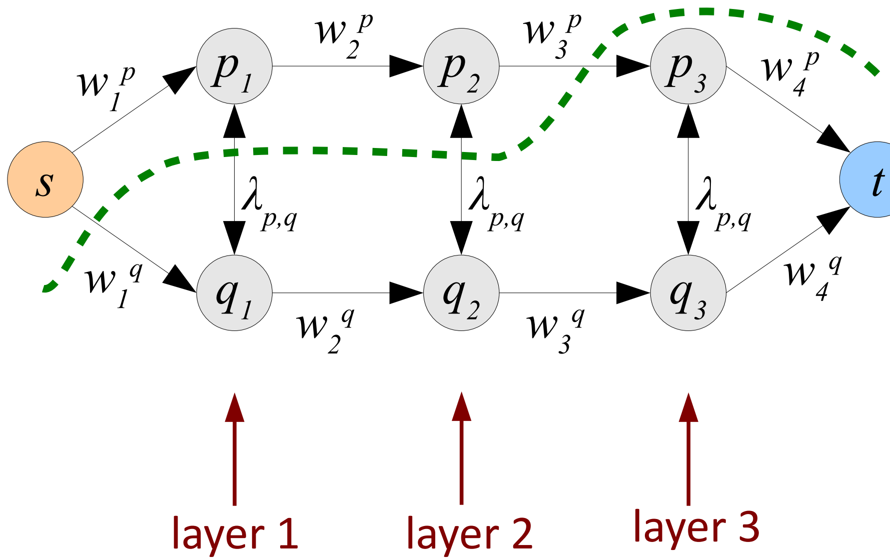
Linear multi-label graph construction

(cf. Boykov et al. 1998)

- Given L $= \{1,\ldots,k\}$
- General idea:
  - construct one layer per label value
  - read label value from cut location
- Any problem ?

e.g., $k = 4$

cut: f_p=3, f_q=1



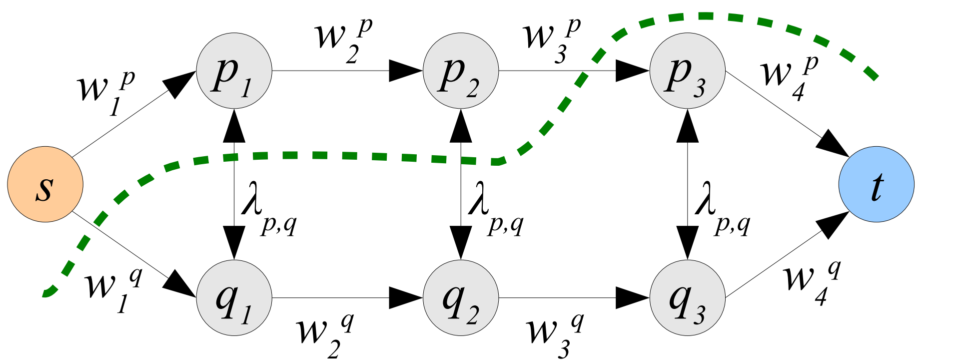
Linear multi-label graph construction

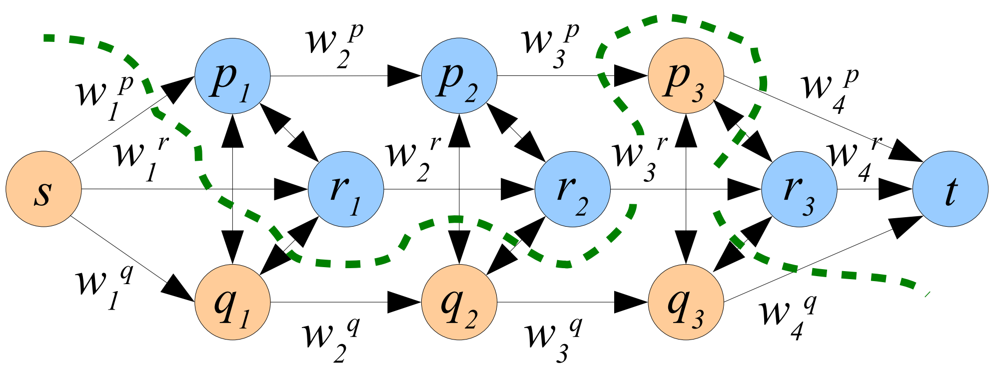
(cf. Boykov et al. 1998)

- Given L $= \{1,\ldots,k\}$
- General idea:
  - construct one layer per label value
  - read label value from cut location
- Any problem ?
  - there could be several cut locations on the same line

e.g., $k = 4$

cut: $f_p=3, f_q=1$



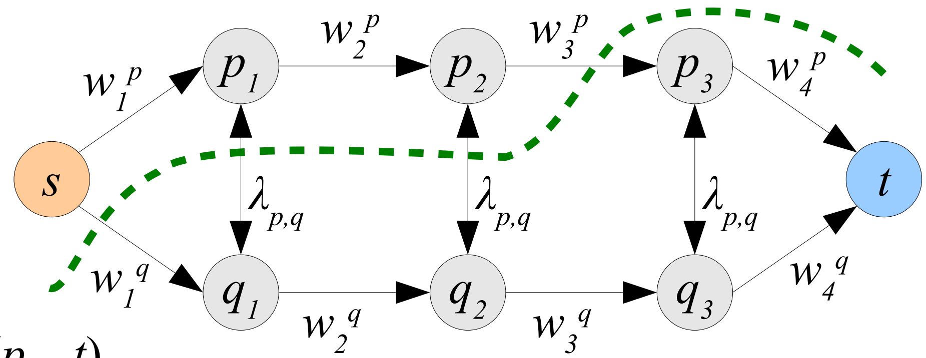


Linear multi-label graph construction

(cf. Boykov et al. 1998)

- $E(f) = \sum_{p\in \mathrm{P}} D_p(f_p) + \sum_{(p,q)\in \mathrm{N}} \lambda_{p,q} |f_p - f_q|$
  with $f_p \in \mathrm{L} = \{1,\ldots,k\}$

cut: $f_p\!=\!3, f_q\!=\!1$



Attempt 2:

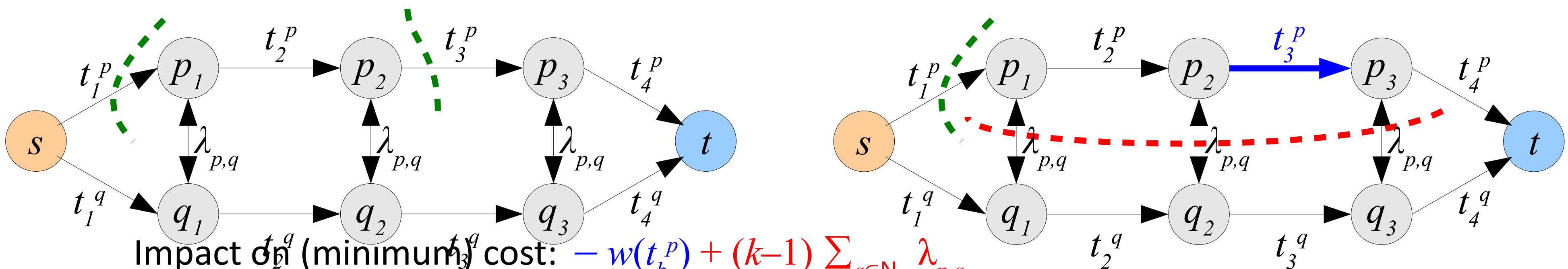
- For each site p
  - create nodes $p_1,\ldots,p_{k-1}$
  - create edges $t_1^p = (s,p_1)$, $t_j^p = (p_{j\text{-}1},p_j)$, $t_k^p = (p_{k\text{-}1},t)$
  - assign weights $w_j^p = w(t_j^p) = D_p(j) + K_p$ [penalize more cutting t_j^p]
    with $K_p = 1 + (k\!-\!1) \sum_{q\in \mathrm{N}_p} \lambda_{p,q}$ (where N_p set of neighbors of p)
- For each pair of neighboring sites p and q
  - create edges $(p_j,q_j)_{j\in\{1,\ldots,k-1\}}$ with weight $\lambda_{p,q}$

Linear multi-label graph properties

(cf. Boykov et al. 1998)

- <u>Lemma:</u> for each site p, a minimum cut severs exactly one t_j^p
  - $[\geq 1]$ Any cut severs at least one t_j^p
  - $[\leq 1]$ Suppose t_a^p, t_b^p are cut (same line p), then build new cut with t_b^p restored and links $(p_j,q_j)_{j\in\{1,\ldots,k-1\}}$ broken for $q \in \mathrm{N}_p$

severed = coupé, sectionné



Impact on (minimum) cost: $-\,w(t_b^p) + (k-1)\sum_{q\in \mathrm{N}_p} \lambda_{p,q}$

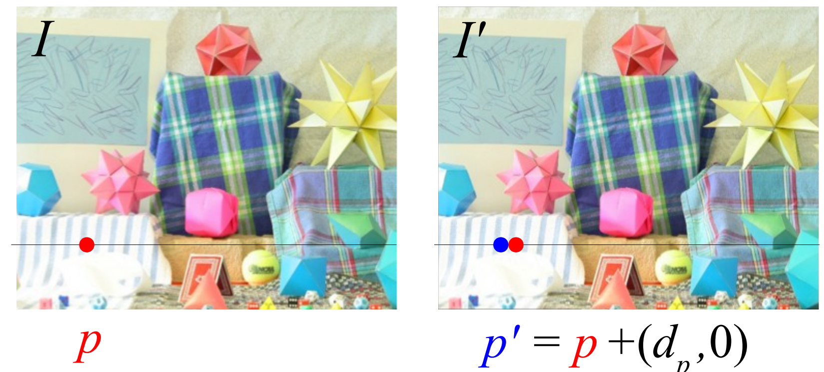
$= -D_p(j) - 1 < 0$ → strictly smaller cost → contradiction

- <u>Theorem</u> (Boykov et al. 1998): a minimum cut minimizes $E(f)$

Application to stereovision: disparity map estimation

rectified images ↔ aligned cameras

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$



- Graph-cut setting
  - discrete disparities: $d_p \in \mathrm{L} = \{d_{\min},\dots,d_{\max}\}$
  - data term: $D_p(d_p)$
    - small when pixel p in I similar to pixel $p' = p + (d_p, 0)$ in I'
  - smoothness term: $V_{p,q}(d_p\,,\, d_q)$
    - small when disparities d_p and d_q are similar

Application to stereovision: disparity map estimation

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$
- Graph-cut setting
  - discrete disparities: $d_p \in \mathrm{L} = \{d_{\min},\ldots,d_{\max}\}$
  - data term: $D_p(d_p)$
    - small when pixel p in I similar to pixel $p' = p + (d_p,0)$ in I'
  - smoothness term: $V_{p,q}(d_p, d_q)$
    - small when disparities d_p and d_q are similar

e.g., what definition?

e.g., what definition?

p

$p' = p + (d_p,0)$

Application to stereovision: disparity map estimation

$\bar{I}_P = 1/|P| \sum_{q \in P} I_q$ $\sigma = [1/|P| \sum_{q \in P} (I_q - \bar{I}_P)^2]^{1/2}$

$E_{ZNSSD}(P\,;\boldsymbol{u}) = 1/|P| \sum_{q \in P} [(I'_{q+\boldsymbol{u}} - \bar{I}'_P)/\sigma' - (I_q - \bar{I}_P)/\sigma]^2$

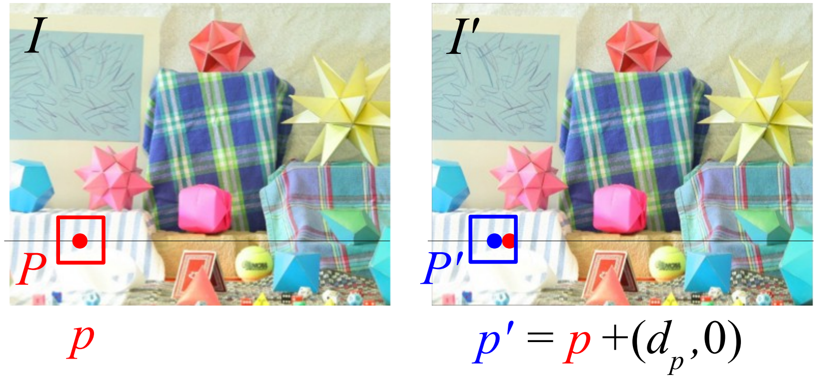
p $p' = p + (d_p, 0)$

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$
- Graph-cut setting
  - discrete disparities: $d_p \in \mathrm{L} = \{d_{\min},\dots,d_{\max}\}$
  - data term: $D_p(d_p)$ — e.g., what definition?
    - e.g., $D_p(d_p) = E_{ZNSSD}(P_p\,;(d_p,0))$ where P_p patch around pixel p
  - smoothness term: $V_{p,q}(d_p\,,\,d_q)$ — e.g., what definition?
    - e.g., $V_{p,q}(d_p\,,\,d_q) = \lambda\,|d_p - d_q|$ [Boykov et al. → optimal disparities]

SSD = sum of square differences
NSSD = normalized ...
ZNSSD = zero-normalized ...

Application to stereovision: disparity map estimation

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$
- Graph-cut setting
  - discrete disparities: $d_p \in \mathrm{L} = \{d_{\min},\ldots,d_{\max}\}$
  - data term: $D_p(d_p)$
    - e.g., $D_p(d_p) = E_{ZNSSD}(P_p; (d_p, 0))$
  - smoothness term: $V_{p,q}(d_p\,,\, d_q)$
    - e.g., $V_{p,q}(d_p\,,\, d_q) = \lambda\,|\,d_p - d_q\,|$ [Boykov et al. → optimal disparities]



Is it the "optimal" solution to the disparity map estimation problem ?

Application to stereovision: disparity map estimation

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$
- Graph-cut setting
  - discrete disparities: $d_p \in \mathrm{L} = \{d_{\mathrm{min}},\ldots,d_{\mathrm{max}}\}$
  - data term: $D_p(d_p)$
    - e.g., $D_p(d_p) = E_{ZNSSD}(P_p ; (d_p, 0))$
  - smoothness term: $V_{p,q}(d_p\,,\, d_q)$
    - e.g., $V_{p,q}(d_p\,,\, d_q) = \lambda\,|\,d_p - d_q\,|$ [Boykov et al. → optimal disparities]

p

$p' = p + (d_p, 0)$

- Meaningful but arbitrary choices: patch size, similarity, smoothness...
- Optimal solution for energy ⇒ optimal solution for problem /

Application to stereovision: disparity map estimation

CC = cross-correlation
NCC = normalized ...
ZNCC = zero-normalized ...

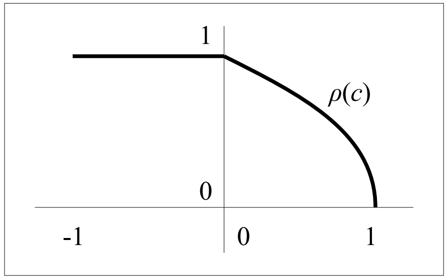
$\bar{I}_P = 1/|P| \sum_{q \in P} I_q$ | $\sigma = [1/|P| \sum_{q \in P} (I_q - \bar{I}_P)^2]^{1/2}$

$E_{ZNCC}(P;\boldsymbol{u}) = 1/|P| \sum_{q \in P} [(I'_{q+\boldsymbol{u}} - \bar{I}'_P)/\sigma' . (I_q - \bar{I}_P)/\sigma]$ | $E_{ZNSSD}(P;\boldsymbol{u}) = 2 - 2\,E_{ZNCC}(P;\boldsymbol{u})$

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$
- Graph-cut setting (alternative)
  - discrete disparities: $d_p \in \mathrm{L} = \{d_{\min},\ldots,d_{\max}\}$
  - $D_p(d_p) = w_{\mathrm{cc}}\, \rho(E_{ZNCC}(P;(d_p,0)))$ with $\rho(c) \in [0,1] \searrow$
    - e.g. $$\rho(c) = \begin{cases} 1 & \text{if } c < 0 \\ \sqrt{1-c} & \text{if } c \geq 0 \end{cases}$$
  - $V_{p,q}(d_p, d_q) = \lambda\, |d_p - d_q|$

p

$p' = p + (d_p, 0)$



dissimilar similar equal

N.B. only $w_{\mathrm{cc}} / \lambda$ matters

Approximate optimization

- Exact multi-label optimization:
  - only limited cases
  - in practice, may require large number of nodes
- How to go beyond exact optimization constraints?

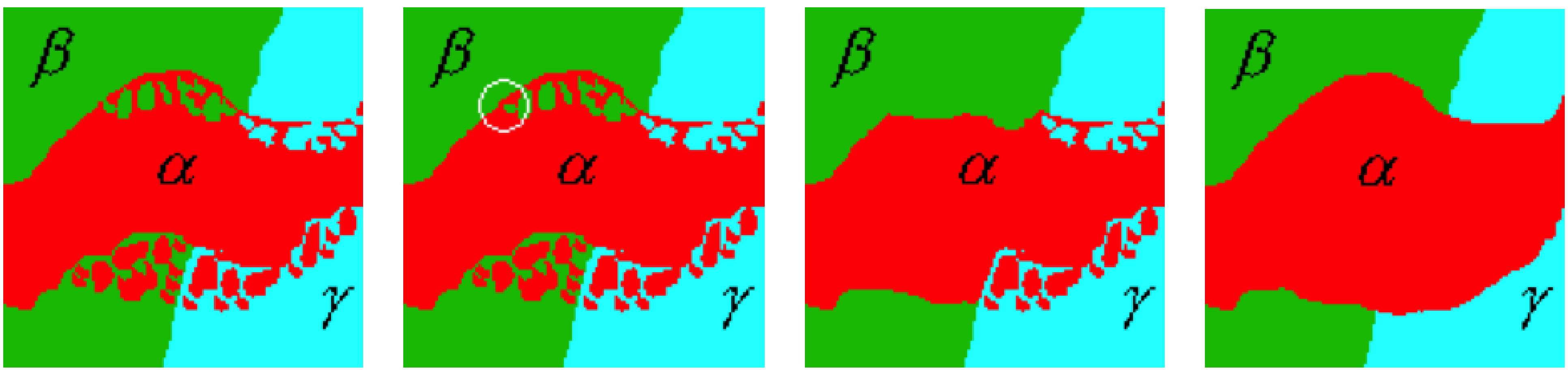
☛ Iterate exact optimizations on subproblems (Boykov et al. 2001)

  - → local minimum ☹
  - but within known bounds of global minimum ☺

Notion of move — Examples

at once = à la fois
move = déplacement
(≈ modification)
de la solution

Move: maps a labeling f : P → L to a labeling f' : P → L
Idea: iteratively apply moves to get closer to optimum f^*



(a) initial labeling

(b) standard move
$\alpha \rightarrow \beta$
at <u>one site</u>
only

(c) α-β-swap
$\alpha \leftrightarrow \beta$
at <u>many sites</u>
at once

(d) α-expansion
any $l \rightarrow \alpha$
at <u>many sites</u>
at once

Moves

move = déplacement (≈ modification) de la solution
α-β-swap = permutation α-β

Given a labeling $f : \mathrm{P} \to \mathrm{L}$ and labels α, β

- f' is a **standard move** from f iff

  f and f' differ at most on one site p

- f' is an **expansion move** (or ***α*-expansion**) from f iff

  $\forall p \in \mathrm{P},\ f'_p = f_p$ or α

  → in f', compared to f, extra sites p can now be labeled α

- f' is a **swap move** (or ***α*-β-swap**) from f iff

  $\forall p \in \mathrm{P},\ f_p \neq \alpha, \beta \;\Rightarrow\; f_p = f'_p$

  → some sites that were labeled α are now β and vice versa

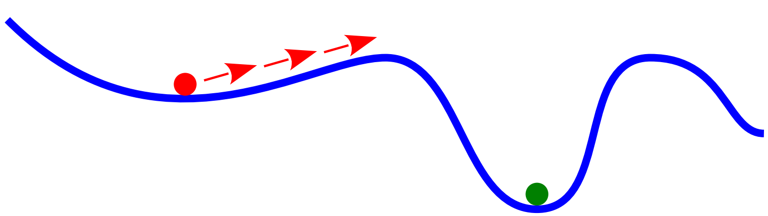
N.B. Other kinds of moves can be defined...

Optimization w.r.t. moves

simulated annealing = recuit simulé
sampling = échantionnage

(cf. Boykov et. al 2001)

- Iterative optimization over moves
  - random cycle over all labels until convergence → local min
- **Iterating standard moves**
  = usual discrete optimization method
  - iterated conditional modes (ICM) = iterative maximization of the probability of each variable conditioned on the rest
    - local minimum w.r.t. standard move,
      i.e., energy cannot decrease with a single pixel label difference
      ⇒ weak condition, low quality
  - simulated annealing, ...
    - slow convergence (optimal properties “at infinity”),
      modest quality, some sampling strategies but mostly random



Optimization w.r.t. moves

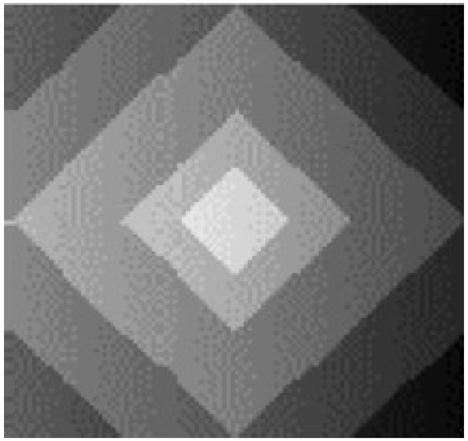
(cf. Boykov et. al 2001)

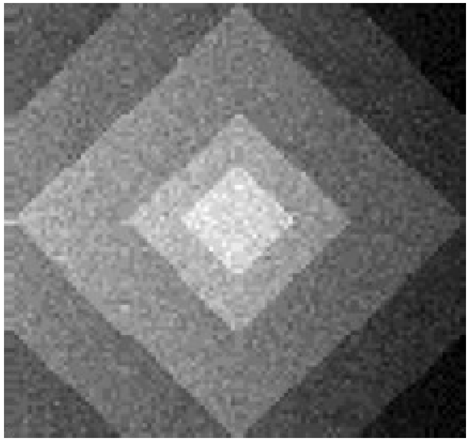
- Iterative optimization over moves
  - random cycle over all labels until convergence → local min
- **Iterating expansion/swap moves** (strong moves)
  - number of possible moves exponential in number of sites
  - compute optimal move using graph cut **= binary problem!**
    - see Boykov et. al 2001 for graph construction and details
  - significantly fewer local minima than with standard moves
  - sometimes within constant factor of global minimum
    - e.g., expansion moves & Potts model → optimum within factor 2

Image restoration with moves

- Restoration with standard moves vs α-expansions

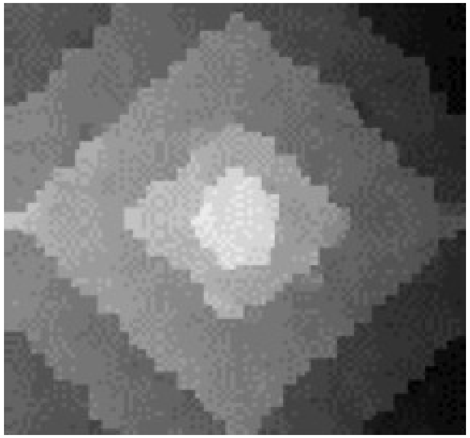
original image

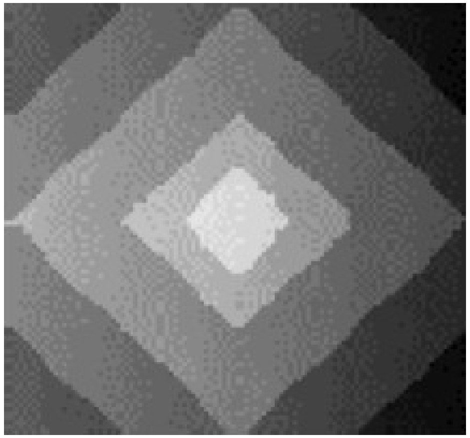




noisy image

restoration with standard moves





restoration with α-expansions

Constraints on interaction potential

(see details in Boykov et. al 2001)

- Expansion move: V metric, → expansion inequality:
  - $V_{p,q}(\alpha,\alpha) + V_{p,q}(\beta,\gamma) \leq V_{p,q}(\beta,\alpha) + V_{p,q}(\alpha,\gamma)$ for all $\alpha,\beta,\gamma \in$ L

metric = métrique
(= fonct distance)
d(x,y) = 0 ⇔ x = y
d(x,y) = d(y,x) ≥ 0
d(x,z) ≤ d(x,y)+d(y,z)

- Swap move: V semi-metric, → swap inequality:

  $V_{p,q}(\alpha,\alpha) + V_{p,q}(\beta,\beta) \leq V_{p,q}(\alpha,\beta) + V_{p,q}(\beta,\alpha)$ for all $\alpha,\beta \in$ L

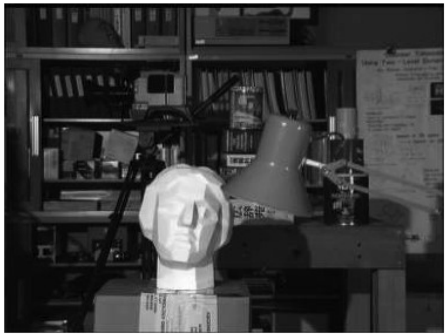
  [= as metric but triangle inequality not required: $V_{p,q}(\alpha,\gamma) \leq V_{p,q}(\alpha,\beta) + V_{p,q}(\beta,\gamma)$]
  [weaker condition than for expansion move]

semi-metric =
semimétrique
d(x,y) = 0 ⇔ x = y
d(x,y) = d(y,x) ≥ 0

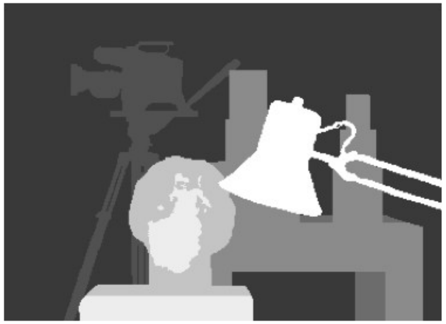
discontinuity-preserving!

- Examples
  - Potts model: $V_{p,q}(\alpha,\beta) = \lambda_{p,q}\, \mathbf{1}(\alpha \neq \beta)$
  - truncated L_2 distance: $V_{p,q}(\alpha,\beta) = \min(K, \|\alpha - \beta\|)$

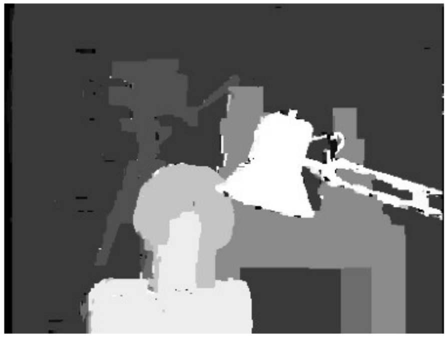
Disparity map estimation with moves



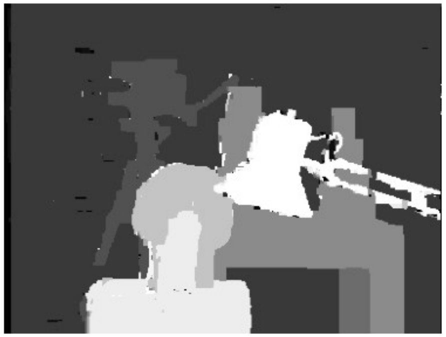
(a) Left image: 384x288, 15 labels



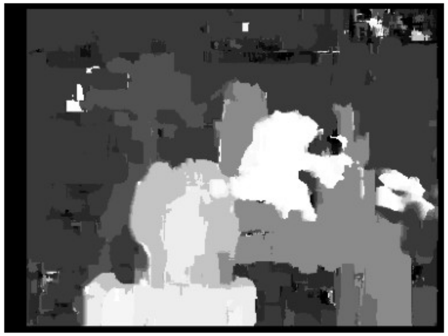
(b) Ground truth



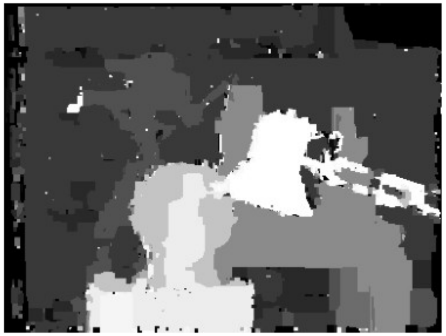
(c) Swap algorithm



(d) Expansion algorithm



(e) Normalized correlation



(f) Simulated annealing

Tsukuba images from famous Middlebury benchmark (also contains Moebius images)

Boykov et al. 2001 © IEEE

To go further on this subject

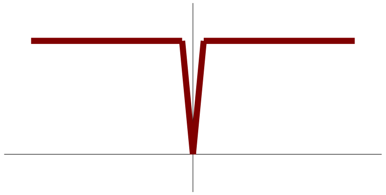
Disparity map estimation: alternative data term

(cf. Boykov et al. 1999, Boykov et al. 2001)

- Idea: direct intensity comparison, but sensitive to sampling
  - $D_p(d_p) = \min(K\,,\ |I_p - I'_{p+d_p}|^2)$
- With image sampling insensitivity:
  - disparity range discretized to 1 pixel accuracy
    → sensitivity to high gradients
  - (sub)pixel dissimilarity measure for greater accuracy, e.g., by linear interpolation (Birchfield & Tomasi 1998)
  - $C_{\text{fwd}}(p,d) = \min_{d-1/2 \le u \le d+1/2} |I_p - I'_{p+u}|$
  - $C_{\text{rev}}(p,d) = \min_{p-1/2 \le x \le p+1/2} |I_x - I'_{p+d}|$ [for symmetry]
  - $D_p(d_p) = C(p,d_p) = \min(K,\ C_{\text{fwd}}(p,d_p),\ C_{\text{rev}}(p,d_p))^2$

No patch similarity here:
the local consistency is given
by the smoothness term

Disparity map estimation: smoothness term

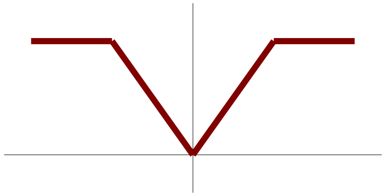


- Scene with fronto-parallel objects
  - **piecewise-constant** model = OK
  - e.g., Potts model:

$$V_{p,q}(d_p,d_q) = u_{p,q}\ \mathbf{1}(d_p \neq d_q)$$

- Scene with slanted surfaces (e.g., ground)
  - **piecewise-smooth** model = better
  - e.g., smooth cap max value:

$$V_{p,q} = \lambda \min(K, |d_p - d_q|)$$

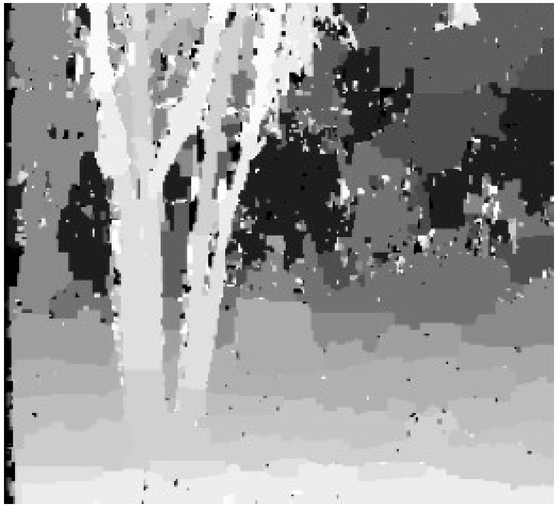


- Metric ⇒ both swap and expansion algorithms usable

Potts model vs smooth cap max value

- Potts model : **piecewise-constant**
  - suited for uniform areas (⇒ fewer disparities on large areas)
- Smooth cap max value: **piecewise-smooth** model
  - suited for slowly-varying areas (e.g., slope)

(a) Left image: 256x233, 29 labels



(b) Piecewise constant model

(c) Piecewise smooth model

To go further on this subject

Disparity map estimation: smoothness term

(cf. Boykov et al. 1998, Boykov et al. 2001)

- Contextual information
  - neighbors p,q more likely to have same disparity if $I_p \approx I_q$
    → make $V_{p,q}(d_p,d_q)$ also depend on $|I_p - I_q|$
  - meaningful in low texture areas (where $|I_p - I_q|$ meaningful)

- E.g., with Potts model: $V_{p,q}(d_p,d_q) = u_{p,q}\ \mathbf{1}(d_p \neq d_q)$
  - $u_{p,q}$: penalty for assigning different disparities to p and q
  - textured regions: $u_{p,q} = K$
  - textureless regions: $u_{p,q} = U(|I_p - I_q|)$
    - $u_{p,q}$ smaller for pixels p,q with large intensity difference $|I_p - I_q|$
    - e.g.,

$$U(|I_p - I_q|) = \begin{cases} 2K & \text{if } |I_p - I_q| \leq 5 \\ K & \text{if } |I_p - I_q| > 5 \end{cases}$$

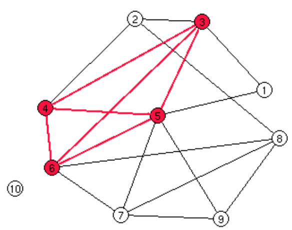
Many extensions to more complex energies

(cf. Pansari & Kumar 2017)

- Truncated Convex Models (TCM)
  - several other approximate algorithms to minimize

$$E(\mathbf{x}) = \sum_{a \in \mathcal{V}} \theta_a(x_a) + \sum_{(a,b) \in \mathcal{E}} \omega_{ab} \min\{d(x_a - x_b), M\}$$

- Truncated Max of Convex Models (TMCM)
  - no clique size restriction (high-order > pairwise)



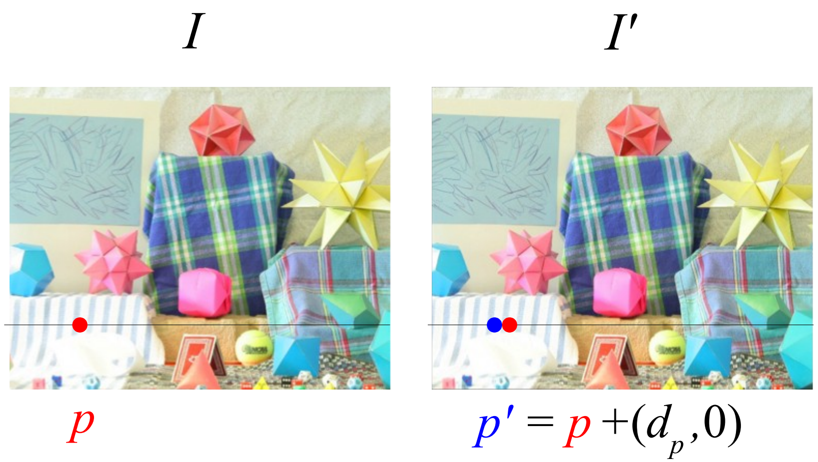
$$\theta_{\mathbf{c}}(\mathbf{x_c}) = \omega_{\mathbf{c}} \sum_{i=1}^{m} \min\{d(p_i(\mathbf{x_c}) - p_{c-i+1}(\mathbf{x_c})), M\}$$

$\mathbf{c}$: clique
$\mathbf{x_c}$: labeling of a clique
$\omega_{\mathbf{c}}$: clique weight
d : convex function
M : truncation factor
$p_i(\mathbf{x_c})$: i-th largest label in $\mathbf{x_c}$
$c = |\mathbf{c}|$

(a) Ground truth (Energy, Time (s)) | (b) Cooccurrence (2098800, 101) | (c) Parsimonious (1364200, 225) | (d) $m = 1$, $h' = 4$ **(1257249, 256)** | (e) $m = 3$, $h' = 4$ (1267449*, 335)

Disparity map estimation

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$
- Are the preceding formulations OK?
  - anything not modeled?
  - any bias?



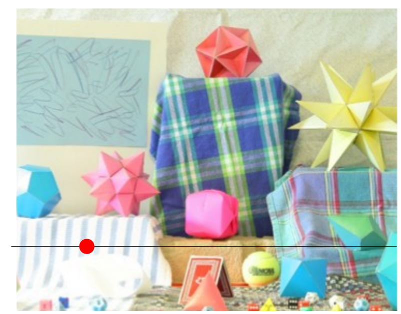
Disparity map estimation

(Boykov et. al 2001)

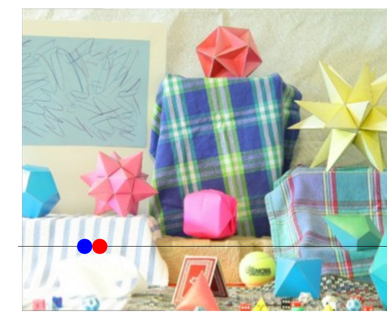
occlusion = occultation

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$

I I'



p



$p' = p + (d_p, 0)$

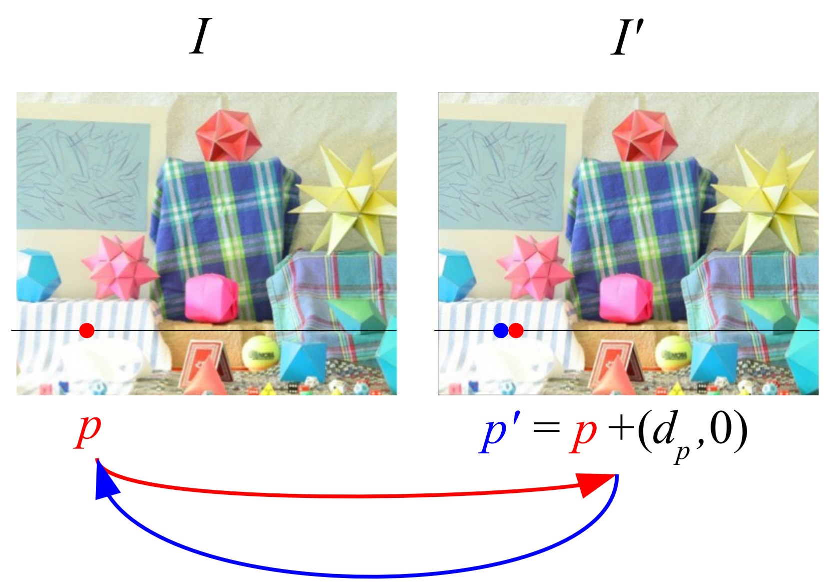
- Are the preceding formulations OK?
  - no treatment of occlusion
  - no symmetry: one center image, one auxiliary image
    - treatment of second image relative to the first (main) one
    - difficulty to incorporate occlusion naturally

Cross-checking

(Bolles & Woodfill, 1993)

occlusion = occultation

- Problem
  - given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for each pixel $p = (u,v)$



- Cross-checking method:
  - compute left-to-right disparity
  - compute right-to-left disparity
  - mark as occlusion pixels in one image mapping to pixels in the other image but which do not map back to them
- Common and easy to implement

Stereovision with occlusion handling

(cf. Kolmogorov & Zabih 2001)

occluded = occulté

- Occlusion
  - pixel visible in one image only
  - occurs usually at discontinuities
- Uniqueness model hypothesis
  - pixel in one image → at most one pixel in other image
    [sometimes too restrictive]
  - pixel with no correspondence: labeled as occluded
- Main idea:
  - use labels representing corresponding pixels (= pixel pairs), not pixel disparity

To go further on this subject

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- A: correspondence candidates (pixel pairs in $I \times I'$) = pixel assignments
  - $\mathsf{A} = \{\, (p,p') \mid p_y = p'_y$ and $0 \leq p'_x - p_x < k \,\}$ (same line, different position)
  - disparity: for $a = (p,p') \in \mathsf{A}$, $d(a) = p'_x - p_x$
  - hypothesis: disparities lie in limited range $[0,k]$
  - goal: find subset of A containing only corresponding pixels
  - use: subsets defined as labelings $f : \mathsf{A} \rightarrow \mathsf{L} = \{0,1\}$ such that $\forall a = (p,p') \in \mathsf{A}$, $f_a = 1$ if p and p' correspond, otherwise $f_a = 0$
  - symmetric treatment of images (& applicable to non-aligned cameras)
- $A(f)$: active assignments, i.e., pixel pairs considered as corresponding
  - $A(f) = \{a \in \mathsf{A} \mid f_a = 1\}$

To go further on this subject

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- $N_p(f)$: set of correspondences for pixel p
  - $N_p(f) = \{a \in A(f) \mid \exists p' \in \mathrm{P},\ a = (p,p')\}$
  - configuration f unique iff $\forall p \in \mathrm{P}\ \ |N_p(f)| \leq 1$
  - occluded pixels defined as pixels such that $|N_p(f)| = 0$
- N : a neighborhood system on assignments (used for smoothness term)
  - $N \subset \{\ \{a_1, a_2\} \subset \mathrm{A}\ \}$
  - for efficient energy minimization via graph cuts:
    - neighbors having the same disparity
    - $N = \{\{(p,p'),(q,q')\} \subset \mathrm{A} \mid p,p'$ are neighbors and $d(p,p') = d(q,q')\}$

      (→ then q,q' are also neighbors)

To go further on this subject

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- $E(f) = E_{\text{data}}(f) + E_{\text{smooth}}(f) + E_{\text{occ}}(f)$
  - $E_{\text{data}}(f) = \sum_{a=(p,p')\in A(f)} (I_p - I'_p)^2$
    - single pixel similarity
  - $E_{\text{smooth}}(f) = \sum_{\{a_1,a_2\}\in \mathrm{N}} V_{a_1,a_2}\, \mathbf{1}(f_{a_1} \neq f_{a_2})$
    - $N = \{\{(p,p'),(q,q')\} \subset \mathrm{A} \mid p,p'$ are neighbors and $d(p,p') = d(q,q')\}$

      → penalty if: $f_{a_1} = 1,\ a_2$ close to $a_1,\ d(a_2) = d(a_1),$ but $f_{a_2} = 0$
    - Potts model **on assignments** (pixel pairs), **not on pixel disparity**
  - $E_{\text{occ}}(f) = \sum_{p\in \mathrm{P}} C_p \,.\, \mathbf{1}(|N_p(f)| = 0)$ [occlusion penalty]
    - penalty C_p if p occluded

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- $E(f) = E_{\text{data}}(f) + E_{\text{smooth}}(f) + E_{\text{occ}}(f)$
- Optimizable by graph cuts as multi-label problem (cf. paper)
  - graph construction on assignments (pixel pairs), not pixels
    - A^α : set of all assignments with disparity α
    - $A^{\alpha,\beta} = A^\alpha \cup A^\beta$
  - expansion move:
    - f' within single α-expansion move of f iff $A(f') \subset A(f) \cup A^\alpha$
      - currently active assignments can be deleted
      - new assignments with disparity α can be added
  - swap move:
    - f' within single swap move of f iff $A(f') \cup A^{\alpha,\beta} = A(f) \cup A^{\alpha,\beta}$
      - only changes: adding or deleting assignments having disparities α or β

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

f unique $\Leftrightarrow$
$\forall p \in \mathrm{P} \quad |N_p(f)| \leq 1$

- Expansion-move algorithm:
  1. start with arbitrary, unique configuration f_0
  2. set success ← false
  3. for each disparity α
     3.1. find $f^\alpha = \text{argmin}_f\, E(f)$
     subject to f unique and within single α-move of f_0
     3.2. if $E(f^\alpha) < E(f_0)$, then set f_0 ← f^α, success ← true
  4. if success go to 2
  5. return f_0
- Critical step: efficient computation of α-move with smallest energy

Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

f unique $\Leftrightarrow$
$\forall p \in \mathcal{P} \ \ |N_p(f)| \leq 1$

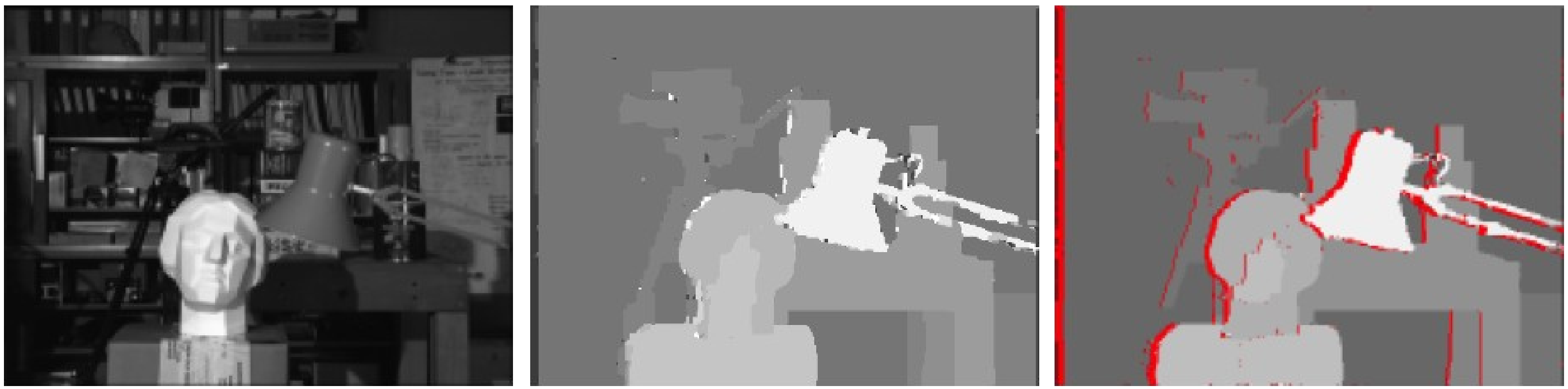
- Swap-move algorithm:
  1. start with arbitrary, unique configuration f_0
  2. set success $\leftarrow$ false
  3. for each pair of disparities α, β $(\alpha \neq \beta)$

     3.1. find $f^{\alpha\beta} = \mathrm{argmin}_f \, E(f)$
     subject to f unique and within single $\alpha\beta$-swap of f_0

     3.2. if $E(f^{\alpha\beta}) < E(f_0)$, then set $f_0 \leftarrow f^{\alpha\beta}$, success $\leftarrow$ true
  4. if success go to 2
  5. return f_0
- Critical step: efficient computation of $\alpha\beta$-swap with smallest energy

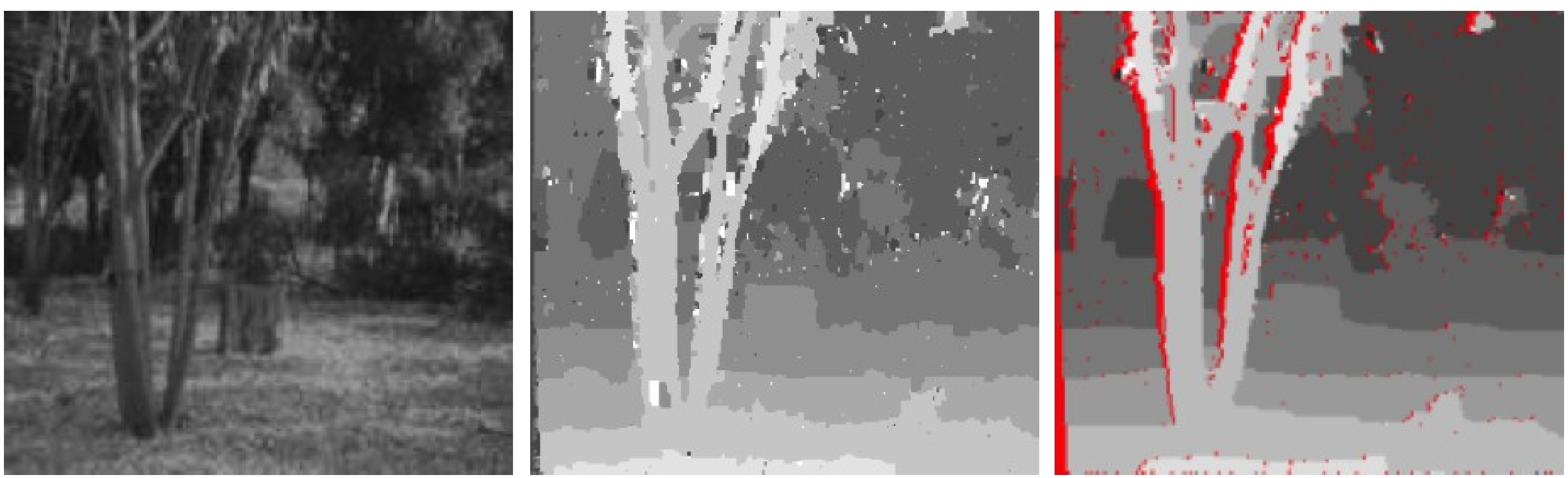
Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)



(a) Left image of *Head* pair (b) Potts model stereo (c) Stereo with occlusions

Disparity maps obtained for the *Head* pair



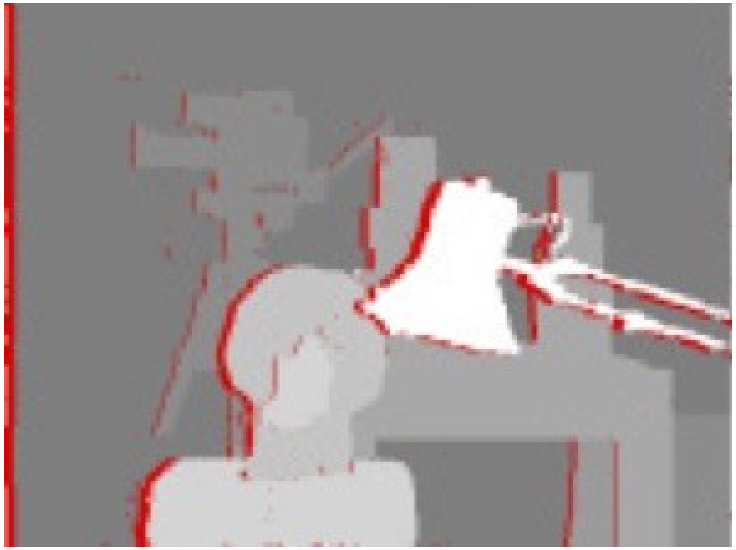
(d) Left image of *Tree* pair (e) Potts model stereo (f) Stereo with occlusions

Disparity maps obtained for the *Tree* pair

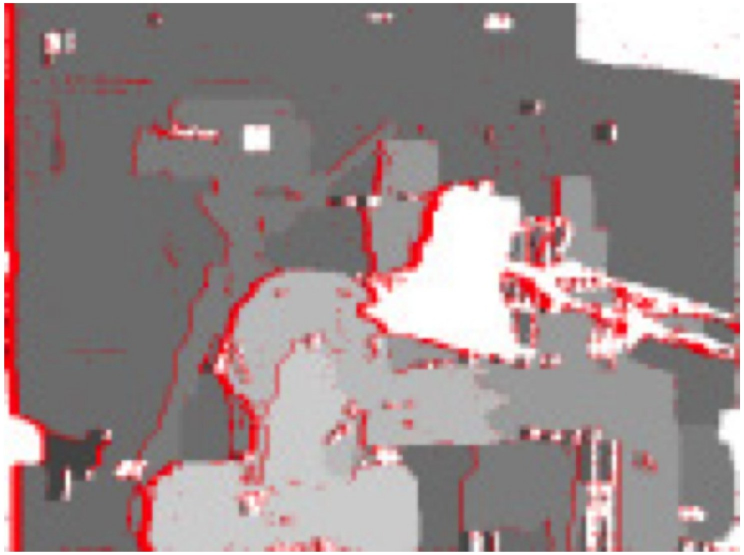
Stereovision with occlusion

(cf. Kolmogorov & Zabih 2001)

- Expansion moves vs swap moves



with α-expansions



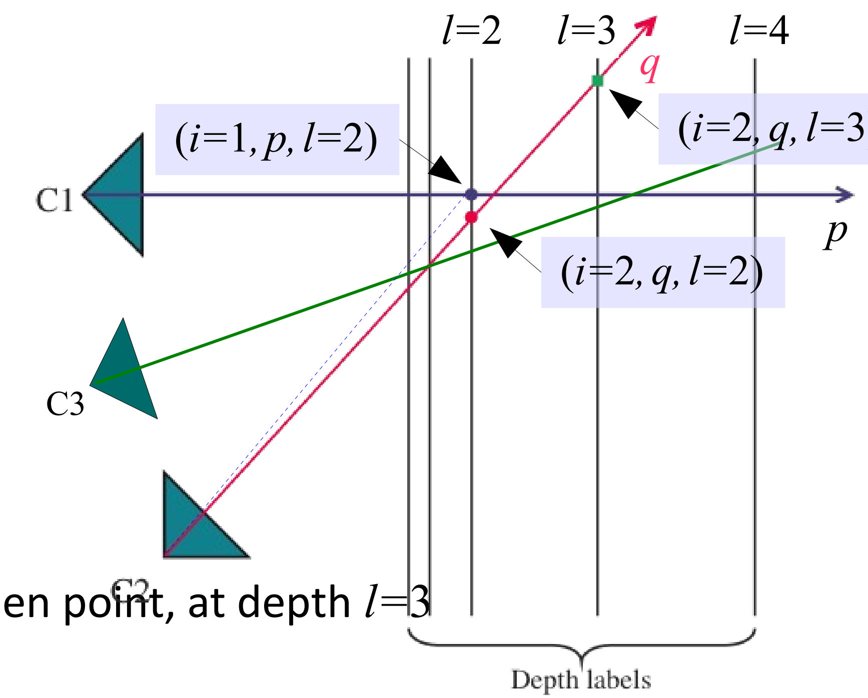
with $\alpha\beta$-swaps

- Swap moves not powerful enough to escape local minima for **this** class of energy function

Multi-view reconstruction

(cf. Kolmogorov & Zabih 2002)

- Given n calibrated images on the "same side" of scene
- Global model
  - L = discretized set of **depths** <u>(not disparities)</u>
  - image i, pixel p, depth l
- Difficulty = point interaction
  - pb: def (i,p,l), (j,q,l) "close" in 3D
    ➔ too many interactions ➔ ☹
  - sol.: def q closest pixel of projection of (i,p,l) on j ➔ ☺
- Photo-consistency constraints (visibility)
  - red point, at depth l=2, blocks C2's view of green point, at depth l=3

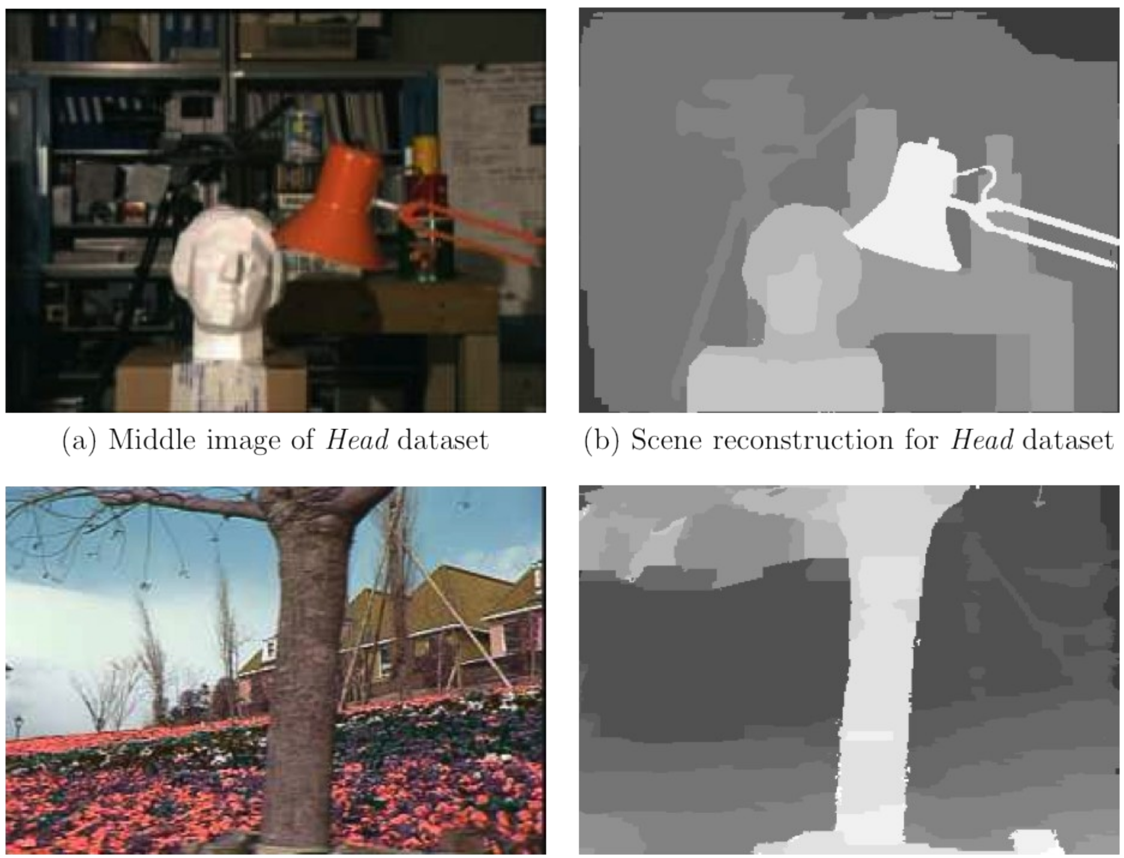


Multi-view reconstruction

(cf. Kolmogorov & Zabih 2002)

- Terms in the energy: data, smoothness, visibility
- Optimization by α-expansion

See paper for details



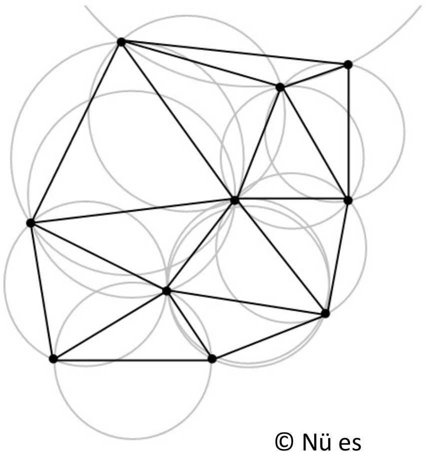
(a) Middle image of *Head* dataset
(b) Scene reconstruction for *Head* dataset
(c) Middle image of *Garden* sequence
(d) Scene reconstruction for *Garden* sequence

point cloud = nuage de points
sweep = balayage
outliers = donnée (ici points) aberrantes
tetrahedralization = tétraédrisation

Beyond disparity maps: 3D mesh reconstruction

(cf. Vu et al. 2012)

- Merging of depth maps into single point cloud
  - possibly sparse depth maps, e.g., obtained by plane sweep
- Problems:
  - multi-view visibility (to be taken into account globally)
  - outliers
- Solution:
  - Delaunay tetrahedralization of point cloud
  - binary labelling of tetrahedra: inside/full or outside/empty
  - 3D surface = interface inside/outside



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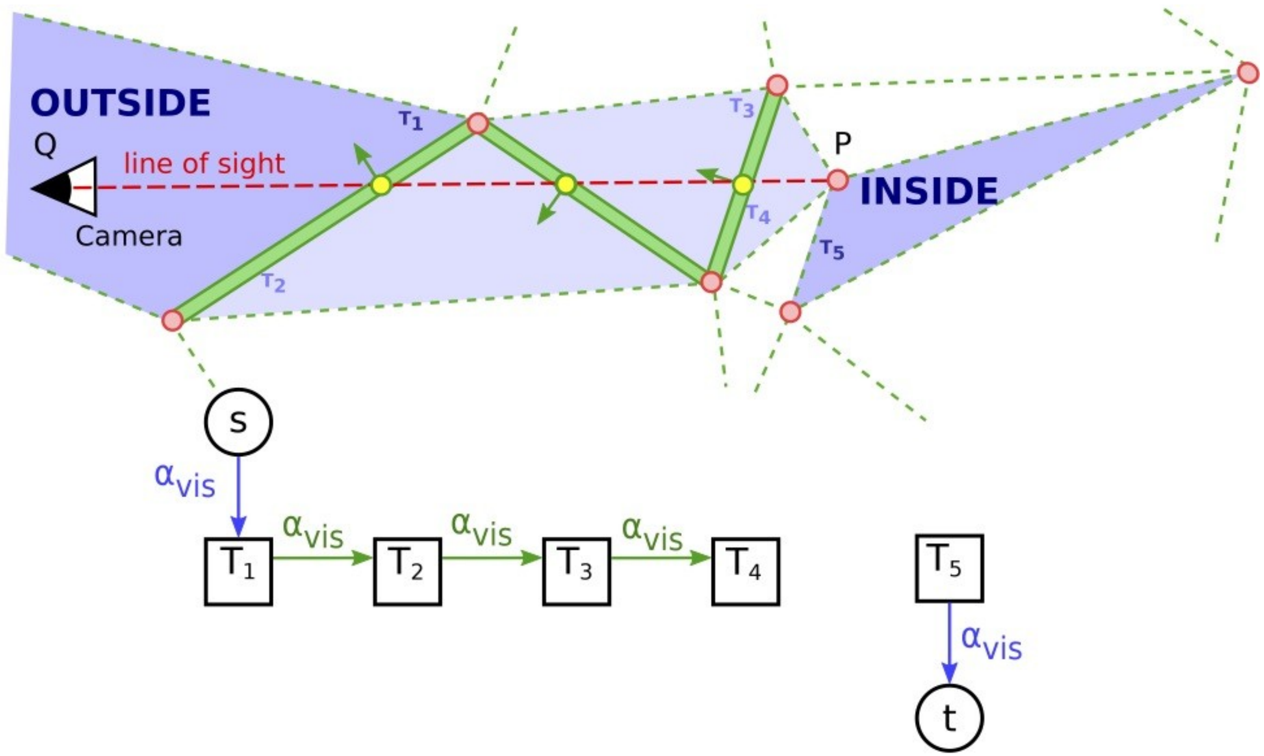
Visibility consistency via graph cut

(cf. Vu et al. 2012)

- Lines of sight from cameras to visible points ⇒ outside

Q,P : points
T : tetrahedron
S : surface
$\boldsymbol{P}$: point cloud
v : line of sight
$l_T = 0$: T outside
(empty space)
$l_T = 1$: T inside
(occupied space)

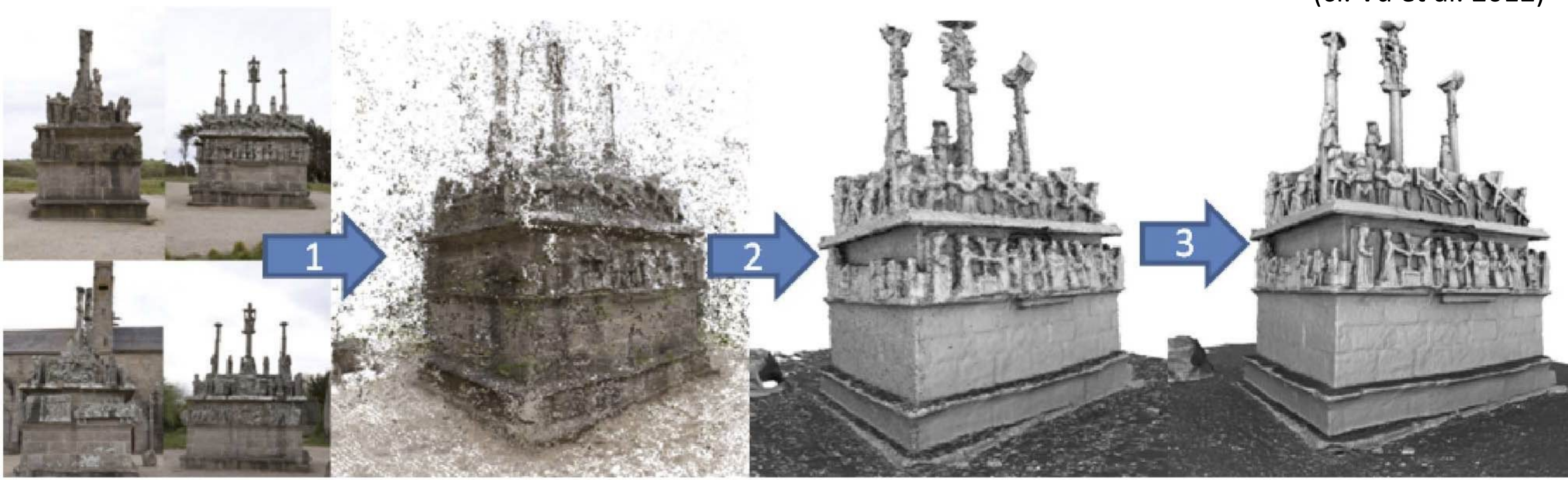


$$D_{\text{out}}(l_T) = \alpha_{\text{vis}} \mathbf{1}[l_T = 1]$$
$$D_{\text{in}}(l_T) = \alpha_{\text{vis}} \mathbf{1}[l_T = 0]$$
$$V_{\text{align}}(l_{T_i}, l_{T_j}) = \alpha_{\text{vis}} \mathbf{1}[l_{T_i} = 0 \wedge l_{T_j} = 1]$$

$$E_{\text{vis}}(S, \boldsymbol{P}, v) = \sum_{P \in \boldsymbol{P}} \left(\sum_{Q \in v_P} D_{\text{out}}(l_{T_1^{Q \to P}}) + \sum_{i=1}^{N_{[PQ]}-1} V_{\text{align}}(l_{T_i^{Q \to P}}, l_{T_{i+1}^{Q \to P}}) + D_{\text{in}}(l_{T_{N_{[PQ]+1}}^{Q \to P}}) \right)$$

Beyond disparity maps: 3D mesh reconstruction

(cf. Vu et al. 2012)

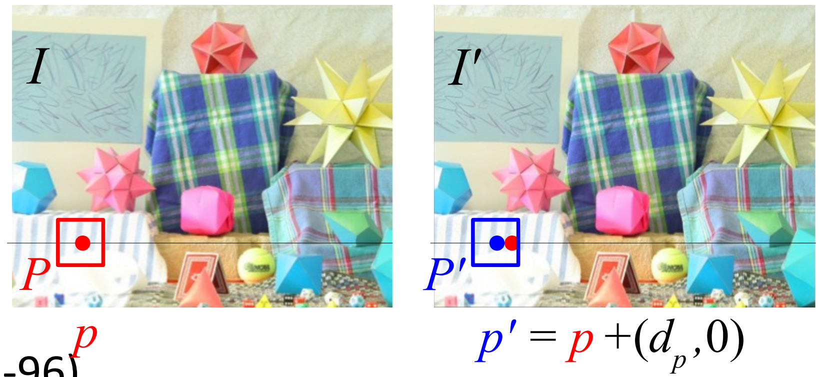


Input images | Point cloud | Visibility-consistent mesh | Refined mesh

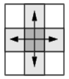
- Best reconstruction results on international benchmarks
- Startup company with IMAGINE members (2011)
  - 15 employees, 90% revenue = international
  - bought by Bentley Systems (2015), still success

acute3D capturingreality

Exercise: simple disparity map estimation (without moves nor occlusion)



- Given 2 rectified images I, I', estimate optimal disparity $d(p) = d_p$ for pixels $p = (u,v)$
- Setting: linear multi-label graph construction (cf. pp. 85-96)
  - discrete disparities: $d_p \in \mathrm{L} = \{d_{\min},\ldots,d_{\max}\}$
  - N_p: 4 neighbors of pixel p
  - $D_p(d_p) = w_{\mathrm{cc}}\ \rho(E_{ZNCC}(P\,;(d_p,0)))$ with
  - $V_{p,q}(d_p\,,\,d_q) = \lambda\,|\,d_p - d_q\,|$



$$\rho(c) = \begin{cases} \dfrac{1}{\sqrt{1-c}} & \text{if } c < 0 \\ & \text{if } c \geq 0 \end{cases}$$

☛ See material provided for the exercise on web site
(template code and detailed exercise description)

N.B. only w_{cc} / λ matters

Advertisement

Internship/PhD positions
related to 3D
in **IMAGINE research group**
(École des Ponts)
and in **Valeo.ai**

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