Vision 3D artificielle
Multiple view geometry

Pascal Monasse pascal.monasse@enpc.fr

IMAGINE, École des Ponts ParisTech

This work is licensed under the Creative Commons Attribution 4.0 International License. To view a copy of this license, visit http://creativecommons.org/licenses/by/4.0/.
Contents

Multi-view constraints

Perspective from $n$ Points

Multi-view calibration
  Incremental calibration
  Global calibration

Methods for Particular Cases
Contents

Multi-view constraints

Perspective from \( n \) Points

Multi-view calibration
  Incremental calibration
  Global calibration

Methods for Particular Cases
Reminder: Triangulation

- Let us write again the binocular formulae:

\[ \lambda x = K(RX + T) \quad \lambda'x' = K'X \]

- Write \( Y^\top = (X^\top \quad 1 \quad -\lambda \quad -\lambda') \):

\[
\begin{pmatrix}
KR & KT & x & 0_3 \\
K' & 0_3 & 0_3 & x'
\end{pmatrix} Y = 0_6
\]

(6 equations \(\leftrightarrow\) 5 unknowns + 1 epipolar constraint)

- We can then recover \( X \).
Multi-linear constraints

- Bilinear constraints: fundamental matrix $x^\top F x' = 0$.
- There are trilinear constraints: $x_i'' = x^\top T_i x'$, which are not combinations of bilinear constraints.
- All constraints involving more than 3 views are combinations of 2- and/or 3-view constraints.
Trilinear constraints

- Write $\lambda_i x_i = K_i (R_i \ T_i) X$
- Write as $AY = 0$ with $Y = (X \ 1 \ -\lambda_1 \ \cdots \ -\lambda_n)^	op$
- Look at the rank of $A$...
Trilinear constraints

Assume $R_1 = Id$ and $T_1 = 0$. We write $A$ of size $3n \times (n + 4)$:

$$A = \begin{pmatrix}
K_1 & 0 & x_1 & 0 & \cdots & 0 \\
K_2 R_2 & K_2 T_2 & 0 & x_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
K_n R_n & K_n T_n & 0 & \cdots & 0 & x_n \\
\end{pmatrix}$$

Subtracting from 3rd column the first column multiplied by $K_1^{-1} x_1$, rank of $A = \text{rank of } A'$ with:

$$A' = \begin{pmatrix}
K_1 & 0 & 0 & 0 & \cdots & 0 \\
K_2 R_2 & K_2 T_2 & -K_2 R_2 K_1^{-1} x_1 & x_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
K_n R_n & K_n T_n & -K_n R_n K_1^{-1} x_1 & \cdots & 0 & x_n \\
\end{pmatrix}$$

Since $K_1$ is invertible, we have to look at the rank of the lower-right $3(n - 1) \times (n + 1)$ submatrix.
Trilinear constraints

\[ \text{Rank of } A = 3 + \text{rank of } B \text{ with } \]
\[
B = \begin{pmatrix}
K_2 T_2 & K_2 R_2 K_1^{-1} x_1 & x_2 & 0 & 0 & \cdots & 0 \\
K_3 T_3 & K_3 R_3 K_1^{-1} x_1 & 0 & x_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
K_{n-1} T_{n-1} & K_{n-1} R_{n-1} K_1^{-1} x_1 & 0 & \cdots & 0 & x_{n-1} & 0 \\
K_n T_n & K_n R_n K_1^{-1} x_1 & 0 & \cdots & 0 & 0 & x_n
\end{pmatrix}
\]

\[ \text{Size of } B: 3(n - 1) \times (n + 1). \]
Trilinear constraints

- $DB$ has same rank as $B$ since $D$ is full rank $3(n - 1)$:

\[
D = \begin{pmatrix}
x_2^T & 0 & 0 & \cdots & 0 \\
0 & x_3^T & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & x_{n-1}^T & 0 \\
0 & \cdots & 0 & 0 & x_n^T \\
[\{x_2\} \times] & 0 & 0 & \cdots & 0 \\
0 & [\{x_3\} \times] & 0 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & [\{x_{n-1}\} \times] & 0 \\
0 & \cdots & 0 & 0 & [\{x_n\} \times]
\end{pmatrix}
\]

- $D$ has size $4(n - 1) \times 3(n - 1)$

- It is easy to check that the kernel of $D$ is $\{0\}$. 
Trilinear constraints

We get:

\[
DB = \begin{pmatrix}
  x_2^\top K_2 T_2 & x_2^\top K_2 R_2 K_1^{-1} x_1 & x_2^\top x_2 & 0 & \cdots & 0 \\
  x_3^\top K_3 T_3 & x_3^\top K_3 R_3 K_1^{-1} x_1 & 0 & x_3^\top x_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
  x_n^\top K_n T_n & x_n^\top K_n R_n K_1^{-1} x_1 & 0 & \cdots & 0 & x_n^\top x_n \\
  M_1 & M_2 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Since \( x_i^\top x_i > 0 \), rank of \( B = (n - 1) + \text{Rank of } M \) (size \( 3(n - 1) \times 2 \)):

\[
M = \begin{pmatrix}
  [x_2] \times K_2 R_2 K_1^{-1} x_1 & [x_2] \times K_2 T_2 \\
  \vdots & \vdots \\
  [x_n] \times K_n R_n K_1^{-1} x_1 & [x_n] \times K_n T_n
\end{pmatrix}
\]

We should have: rank of \( M = 1 \), so that rank of \( A = n + 3 \).

Write that \( 2 \times 2 \) submatrices of \( M \) should have determinant 0.
Trilinear constraints

Proposition Let $M$ a $3n \times 2$ matrix written in blocks of 3 rows:

$$M = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$$

Rank of $M < 2$ iff $\forall i, j$, $a_i b_j^T - b_i a_j^T = 0$. 
Proposition Let $M$ a $3n \times 2$ matrix written in blocks of 3 rows:

$$M = \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix}$$

Rank of $M < 2$ iff $\forall i, j$, $a_i b_j^\top - b_i a_j^\top = 0$.

Proof: The case $b = 0$ is trivial, assume $b \neq 0$.

$\Rightarrow$ We have $a = \lambda b$, and $a_i b_j^\top - b_i a_j^\top = \lambda(b_i b_j^\top - b_i b_j^\top) = 0$.

$\Leftarrow$ We have $(a_i b_j^\top - b_i a_j^\top)^{kl} = a_i^k b_j^l - b_i^k a_j^l = \begin{vmatrix} a_i^k & b_i^k \\ a_j^l & b_j^l \end{vmatrix}$. We get that all $2 \times 2$ submatrices of $M$ have null determinant.
Trilinear constraints

- **Proposition** Let $M$ a $3n \times 2$ matrix written in blocks of 3 rows:

$$
M = \begin{pmatrix}
a_1 & b_1 \\
\vdots & \vdots \\
a_n & b_n
\end{pmatrix}
$$

Rank of $M < 2$ iff $\forall i, j, a_ib_j^\top - b_ia_j^\top = 0$.

- For $i = j$, $([x_i] \times K_i R_i K_1^{-1} x_1) \times ([x_i] \times K_i T_i) = 0$ amounts to

$$
([x_i] \times K_i T_i)^\top (K_i R_i K_1^{-1} x_1) = |x_i \ K_i T_i \ K_i R_i K_1^{-1} x_1| = 0 \text{ or }
$$

$$
x_i^T K_i^{-\top} [T_i] \times R_i K_1^{-1} x_1 = 0 \text{ (epipolar constraint)}
$$

- $[x_i] \times K_i R_i K_1^{-1} x_1 ([x_j] \times K_j T_j)^\top - [x_i] \times K_i T_i ([x_j] \times K_j R_i K_1^{-1} x_1)^\top = 0$

$$
[x_i] \times \left( \sum_k x_1^k T_{ij}^k \right) [x_j] \times = 0 \text{ (9 trilinear constraints)}
$$
Trilinear constraints

- A triplet \((x_1, x_i, x_j)\) imposes at most 4 independent constraints on \(T_{ij}^k\) because of the cross-products.
- Rank of \(M = 0\) (multiple solutions \(X\)) means

\[
\forall i, [x_i] \times K_i R_i K_1^{-1} x_1 = [x_i] \times K_i T_i = 0
\]

so that \(K_i R_i K_1^{-1} x_1\) and \(K_i T_i\) are proportional, hence \(x_1 = \lambda K_1 R_i^\top T_i\) (epipole in image 1 wrt image \(i\)), so that all camera centers are aligned and \(X\) is on this line.
Contents

Multi-view constraints

Perspective from \( n \) Points

Multi-view calibration
  Incremental calibration
  Global calibration

Methods for Particular Cases
**PnP**

- “PnP” = Perspective from $n$ Points.
- From 2D-3D correspondences $(x_i, X_i)$ and known $K$, recover $P = K \begin{pmatrix} R & T \end{pmatrix}$ so that $x_i \sim PX_i$
- Remember calibration from a 3D rig: same problem but with unknown $K$, $P = K \begin{pmatrix} R & T \end{pmatrix}$.
- Minimal problem: $n = 3$, $P3P$ problem, up to 4 solutions:

\[
\begin{align*}
Y^2 + Z^2 - pYZ - a^2 &= 0 \quad (p = 2 \cos \alpha) \\
X^2 + Z^2 - qXZ - b^2 &= 0 \quad (q = 2 \cos \beta) \\
X^2 + Y^2 - rXY - c^2 &= 0 \quad (r = 2 \cos \gamma)
\end{align*}
\]

Write $x = X/Z$, $y = Y/Z$, $a' = a^2/c^2$, $b' = b^2/c^2$, $v = c^2/Z^2$.

\[
\begin{cases}
y^2 + 1 - py - a'v = 0 \\
x^2 + 1 - qx - b'v = 0 \\
x^2 + y^2 - rxy - v = 0
\end{cases} \Rightarrow \begin{cases}
(1 - a')y^2 - a'x^2 + a'rx y - py + 1 = 0 \\
(1 - b')x^2 - b'y^2 + b'rx y - qx + 1 = 0
\end{cases}
\]

(intersection of two conics)
PnP, $n \geq 4$


- Write $X_i = \sum_{j=1\ldots4} \alpha_{ij} C_j^w$ with $C_j^w$ four fixed points.
- Write $C_j = RC_j^w + T$ in camera coordinates.
- Project onto image to obtain a $2n \times 12$ linear system on $\{C_j\}$:

$$[K^{-1}x_i] \times \sum_{j=1\ldots4} \alpha_{ij} C_j = 0$$

- From $MC = 0$, write $C = \sum_{k=1\ldots N} \beta_k V^k$ with the $N$ last columns of $V$ from SVD of $M$
  ($n = 4 \rightarrow N = 4$, $n = 5 \rightarrow N = 2$, $n \geq 6 \rightarrow N = 1$)
- Write the conservation of distances (6 equations in $\beta$):

$$1 \leq i < j \leq 4 : \left\| \sum_{k=1\ldots N} \beta_k (V^k_{3i-2:3i} - V^k_{3j-2:3j}) \right\| = \|C_i^w - C_j^w\|$$
Contents

Multi-view constraints

Perspective from $n$ Points

**Multi-view calibration**
  * Incremental calibration
  * Global calibration

Methods for Particular Cases
Incremental multi-view calibration

1. Compute two-view correspondences
2. Build tracks (multi-view correspondences)
3. Start from initial pair: compute $F$, deduce $R$, $T$ and 3D points (known $K$)
4. Add image with common points.
5. Estimate pose ($R$, $T$)
6. Add new 3D points
7. Bundle adjustment
8. Go to 4

See open source software Bundler: SfM for Unordered Image Collections (http://www.cs.cornell.edu/~snavely/bundler/)
Incremental multi-view calibration

$X_i$

$O_L$

Left view

$O_R$

$X_{1i}$

$X_{2i}$

$P = ? = K[R|t]$
Incremental multi-view calibration
Bundle adjustment

We have the equations

\[ x_{ij} = DK \left( R_j \  T_j \right) X_i \]

- \( i \): 3D point index
- \( j \): view index
- \( x_{ij} \): 2D projection in view \( j \) of point \( X_i \)
- \( D \): geometric distortion model
- \( K \): internal parameters of camera

Minimize by Levenberg-Marquardt the error

\[ E = \sum_{ij} d(x_{ij}, DK \left( R_j \  T_j \right) X_i)^2 \]
Global calibration

- Compute $E_{ij}$, essential matrices between all views $i$ and $j$
- Extract $R_{ij}$ and $T_{ij}$ from $E_{ij}$
- Rotation alignment: recover $\{R_i\}$, global rotation with respect to $R_0 = Id$, such that $R_i = R_{ij}R_j$ for all $i, j$
  - [Martinec&Pajdla CVPR 2007]: write $R_{ij}$ as unitary quaternions, find minimum of
    \[
    \sum_{ij} \| q_i - q_{ij}q_j \|^2 \quad \text{with} \quad \| (q_1^\top \cdots q_n^\top)^\top \| = n
    \]
    But this does not ensure $\|q_i\| = 1$, condition for a quaternion to represent a rotation...
  - No exact solution since $R_{ij}$ can have an error. How to close loops?
  - What about outliers among the $R_{ij}$?
- Translation alignment: recover $\{T_i\}$, global translation with respect to $T_0 = 0$, such that $T_i = R_{ij}T_j + \lambda_{ij}T_{ij}$
Global calibration

- Compute $E_{ij}$, essential matrices between all views $i$ and $j$
- Extract $R_{ij}$ and $T_{ij}$ from $E_{ij}$
- **Rotation alignment**: recover $\{R_i\}$, global rotation with respect to $R_0 = Id$, such that $R_i = R_{ij}R_j$ for all $i, j$
- **Translation alignment**: recover $\{T_i\}$, global translation with respect to $T_0 = 0$, such that $T_i = R_{ij}T_j + \lambda_{ij}T_{ij}$
  - [Moulon&Monasse&Marlet ICCV 2013]: solve the linear programme

$$\min_{\{T_i\},\{\lambda_{ij}\},\gamma} \gamma \text{ with } \lambda_{ij} \geq 1, \|T_i - R_{ij}T_j - \lambda_{ij}T_{ij}\|_\infty \leq \gamma$$
Some results

Orangerie dataset
Some results

Opera Garnier dataset
Contents

Multi-view constraints

Perspective from \( n \) Points

Multi-view calibration
  Incremental calibration
  Global calibration

Methods for Particular Cases
Visual hull

- We assume we are able to segment the object of interest in each view
- From the silhouette, we can restrict the location inside a cone
- Intersect cones from all views
- The result is called the **visual hull**

Epipolar plane imagery

A technique for depth estimation from a movie with controlled motion

- Assume a uniform motion of camera along the horizontal line
- Consider 2D cuts \((x, y^*, t)\) of the volume
- Edges move along lines, whose slope is the disparity
- Advantage: large baseline between distant time steps (accurate estimation) and small baseline between close time steps (easier tracking)

Source: http://www.informatik.uni-konstanz.de/cvia/research/light-field-analysis/consistent-depth-estimation/
Software

Infrastructure:

▶ *Eigen*: C++ library for linear algebra
▶ Google’s *Ceres Solver* for bundle adjustment (automatic differentiation)

SfM pipelines:

▶ *Bundler* (2008, open source, University of Washington)
▶ *PhotoScan* (2010, commercial, Agisoft)
▶ *VisualSfM* (2011, open source, University of Washington): GPU
▶ *OpenMVG* (2012, open source, École des Ponts ParisTech)
▶ *ColMap* (2016, open source, University of North Carolina)
Conclusion

▶ Multi-view reconstruction is an active and lively field of research, but less explored than 2-view stereo correspondence
▶ Project: openMVG (incremental and global pipelines)