1 Computation of homography

In the computation of a homography from (finite) point matches, we can write the vector equation \( X'_i \times (HX_i) = 0 \) as two independent linear equations in the parameters \( h = (H_{11} \ H_{12} \ldots H_{33}) \) of \( H \):

\[
\begin{pmatrix}
  x_i & y_i & 1 & 0 & 0 & -x'_i x_i & -x'_i y_i & -x'_i \\
  0 & 0 & 0 & x_i & y_i & 1 & -y'_i x_i & -y'_i y_i & -y'_i 
\end{pmatrix}
\begin{pmatrix}
  h_1 \\
  h_2 \\
  \vdots \\
  h_9 
\end{pmatrix}
= 0.
\]

From 4 point matches, we get an \( 8 \times 9 \) system in \( h \), so \( h \) is a non-null vector of the kernel of the stacked matrix \( A \). There seems to be always a unique solution but obviously there are degenerate situations. The following questions examine these degeneracies.

1. Explain why there is no solution if \( X_1 = X_2 \) and points \( X'_1 \) and \( X'_2 \) are distinct.

2. By examining the structure of the matrix \( A \), explain why there is no unique solution if the points \( X_1, X_2 \) and \( X_3 \) are distinct but aligned.

3. Show the form of the homography matrix \( H \) keeping fixed all points of line \( y = 0 \) and also the point \((0, 1)\) of Euclidean plane. Explain why it is a 1-parameter family of homographies.

2 Sampson error for fundamental matrix

We recall that the Sampson error \( \epsilon_S(X) \) for a given geometric model, an approximation of the geometric error based on the algebraic error \( \epsilon(X) \), can be written:

\[
\epsilon_S = \epsilon^T (JJ^T)^{-1} \epsilon
\]
with \( J \) the matrix of partial derivatives (Jacobian matrix) of \( \epsilon \) as a function of data point \( X \). This exercise computes the formula of Sampson error for fundamental matrix \( F \).

The algebraic error is \( P^T F P = 0 \) with \( P = (x \ y \ 1)^T \) and \( P' = (x' \ y' \ 1)^T \).

1. Compute \( J \) at point \( X = (x \ y \ x' \ y') \).

2. Write formula for Sampson error associated to \( F \).

3. Discuss its relation to geometric error.

### 3 Recovery of projection matrices from \( F \)

From \( P \) and \( P' \), 3 \( \times \) 4 projection matrices in left and right images, we can compute \( F \), the fundamental matrix. The goal of this exercise is to recover all possible pairs of matrices \((P,P')\) from a given \( F \). We note \( e' \) the epipole in right image \((F^T e' = 0)\).

1. Show that if \( P = (I|0) \) and \( P' = ([e']_\times F|x') \), then \( S = P'^T F P \) is skew-symmetric \((X^T S X = 0 \text{ for all } X)\).

2. Deduce that the above pair \((P,P')\) gives rise to \( F \). Why is the right camera of this particular solution not a pinhole camera?

3. Show that if \( P = (I|0) \) and \( P' = (A|a) \) (with \( A \) a 3 \( \times \) 3 matrix and \( a \) a 3-vector), then \( F = [a]_\times A \).

4. Suppose \( F = [a]_\times A = [a']_\times A' \). Prove that there is some \( \lambda \neq 0 \) and some 3-vector \( v \) such that
   \[
   a' = \lambda a \text{ and } \lambda A' = A + av^T. 
   \]

5. Prove that if \( P = (I|0) \), then \( P' = ([e']_\times F + e'v^T|\lambda e') \) with \( \lambda \neq 0 \) and \( v \) a 3-vector.

6. Explain why if the pair \((P,P')\) yields \( F \) then so does \((PH,P'H)\) for any 4 \( \times \) 4 matrix \( H \), then show that the general solution is such that \( P = (P_{3\times3}|P_4) \) with \( P_{3\times3} \) invertible, and
   \[
   P' = ([e']_\times F + e'v^T)P + e'w^T
   \]
   with \( w \) a 4-vector such that
   \[
   \begin{pmatrix} P \\ w^T \end{pmatrix}
   \]
   be invertible.
4 Feature detection and description

1. Between a 90° corner and a 45° corner, which one has the strongest response according to the Harris-Stephens criterion, and why?

To answer this question, you may simply compare the case of the central point in images \( I \) and \( I' \) defined by:

\[
I = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \quad \quad I' = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

As indicated in the lecture on the Harris corner detector, directional derivatives can be computed convoluting respectively with \([-1 0 1]\) and \([-1 0 1]^T\).

You may restrict yourself to a derivative Gaussian, and forget about the integration Gaussian. To simplify calculations, you may also use an approximate Gaussian kernel defined as:

\[
g = \frac{1}{16} \begin{bmatrix}
1 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 1
\end{bmatrix}
\]

2. Does the answer differ if you consider the corner strength as defined by Brown et al. (2005), i.e., the harmonic mean of the eigenvalues?

3. Let \( x_1, x_2, x_3 \) be detected SIFT points in image \( I \), \( x'_1 \) be a detected SIFT points in image \( I' \) and \( v_1, v_2, v_3, v'_1 \) be the corresponding SIFT descriptors. Assume that \( \|v_1 - v_2\| = 0.1 \) and \( \|v_1 - v_3\| = 0.11 \). What can be said of a possible match between \( x_1 \) and \( x'_1 \)?

5 Graph cuts

1. What is the minimal cut of the following graph, and why?
2. When using a graph cut for estimating a disparity map as explained in the course, suppose you use a zero-mean normalized sum of square differences ($E_{ZNSSD}$) rather than a zero-mean normalized cross correlation ($E_{ZNCC}$). Has anything to be changed in edge weight definition?