1 Transfer of epipolar lines

We consider a stereo image pair linked by the epipolar constraint $x^T F x' = 0$ with epipoles $e_L$ and $e_R$: $F^T e_L = F e_R = 0$.

1. Remind the definition of an epipolar line.

2. Explain geometrically why for any point $x$ on a given left epipolar line $\ell_L$, its homologous point is on a right epipolar line $\ell_R$ independent on the choice of $x$. This shows there is a function $H$ mapping left epipolar lines $\ell_L$ to their corresponding right epipolar line $\ell_R$. We show in the next questions that $H$ is not unique, but that most are homographies:

   \[
   H(v) = F^T [e_L]_\times + v e_L^T, \tag{1}
   \]

   with $v \in \mathbb{R}^3$.

3. Show that $x = e_L \times \ell_L$ is a point on epipolar line $\ell_L$. Using this point, show that for any $v$ in (1), $H(v)\ell_L = \ell_R$.

4. Let $H_0$ be a homography mapping left epipolar lines to the corresponding right epipolar lines. Show that there is some scalar $\lambda$ such that

   \[
   \lambda F^T = H_0[e_L]_\times. \tag{2}
   \]

   (Consider the mapping of any $x \in \mathbb{R}^3$ through each member of this equation)

5. Show then that up to scale

   \[
   H_0 = H(-\frac{1}{\lambda} H_0 e_L). \tag{3}
   \]

   The remaining questions prove that $H(v)$ is a homography if and only if $v^T e_R \neq 0$.

6. Consider two independent vectors $v_1$ and $v_2$ orthogonal to $e_L$. Show that $e_L \times v_1$ and $e_L \times v_2$ are two distinct left epipolar lines.

7. Deduce that $H(v)v_1$ and $H(v)v_2$ are two distinct right epipolar lines.

8. Show that $H(v)v_1 \times H(v)v_2 = e_R$ up to scale.

9. Show that $H(v)e_L$ is in the span of vectors $H(v)v_1$ and $H(v)v_2$ if and only if $v^T e_R = 0$.

10. Show that $H(v)$ is a homography if and only if $(H(v)e_L, H(v)v_1, H(v)v_2)$ is a basis.

11. Conclude.
2 Pose estimation with relative depth priors

Some magical deep-learning based methods can achieve the impossible: infer the depth of pixels in a single image, up to an unknown global scale $s$.

12. Writing $p_1 = (x_1, y_1, 1)^T$ and $p_2 = (x_2, y_2, 1)^T$ matching points in a stereo pair, with relative depth $d_1$ and $d_2$ (up to scale factors $s_1$ and $s_2$), explain why they are linked through

$$s_2 d_2 p_2 = s_1 d_1 R p_1 + t,$$

with $R$ and $t$ the relative rotation and translation between the two views.

13. Write 3 linear equations satisfied by the following 13 coefficients of a vector $x$ composed of: 9 unknowns from $s_1 R$, 3 from $t$ and $s_2$.

14. Assuming 4 or more matching pairs $(p_1, p_2)$ are known, what is the procedure to recover $R$, $t$, $s_1$ and $s_2$?

15. Show that we can write six quadratic polynomial constraint equations involving the coefficients of a $3 \times 3$ matrix to be of the form $s_1 R$.

16. Assuming only two matching pairs $(p_1, p_2)$ are given, explain how $x$ can be written as a linear combination of 7 orthonormal vectors that we can compute. Show that in theory we are able to recover $x$ up to some scale.\footnote{More precisely, $x$ is among a set of up to 16 possible solutions, up to scale.}

17. Assume that we have only relative depth prior for the first image, not the second one, and that three matching pairs $(p_1, p_2)$ are available.

   (a) For each pair, write two linear equations involving $y$, the vector composed of the 12 first coefficients of $x$ (excluding $s_2$).

   (b) Explain how $y$ can be written as a linear combination of 6 orthonormal vectors that we can compute.

   (c) Show that in theory we are able to recover $y$ up to some scale.\footnote{In this case, there are up to 8 solutions.}

18. Suppose we know three pairs $(p_1^i, p_2^i)$ with their relative depths $d_1^i$ and $d_2^i$.

   (a) Justify that points on the plane spanned by $\{p_1^i\}$ are related to points on the plane spanned by $\{p_2^i\}$ through a homography $H$.

   (b) Show the centroid of $\{p_1^i\}$ is mapped by $H$ to the centroid of $\{p_2^i\}$.

   (c) Deduce that the homography $H$ can be computed.

19. For each of the above situations, propose an algorithm to recover $(R, t)$ or $H$ from the stereo pair with the estimated relative depths. How is the relative depth useful and how can it lead to faster estimation?