3D Computer Vision - MVA final exam
(duration: 2h30)

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November 26, 2024

1 Structure and Motion from Two Views of a Plane

1.1 Preliminaries

1. We assume that we have two views of a plane from calibrated cameras (known K and K'). The goal is to recover the relative poses of the cameras (motion) and the plane position (structure). In what follows, we assume $K = K' = I$. Justify that there is no loss of generality with this assumption.

2. Show that in general we can write the plane equation as:
$$n^\top X = 1, \tag{1}$$
with $n, X \in \mathbb{R}^3$.

3. Placing the camera center coordinates at left view, show that we can write the projection equations (in homogeneous coordinates) of a 3D point X as:
$$x = X \qquad\qquad x' = R(X - C). \tag{2}$$
What do $R \in \mathbb{R}^{3\times3}$ and $C \in \mathbb{R}^3$ represent geometrically?

4. Show that we have
$$x' = Hx \text{ with } H = R(I - Cn^\top). \tag{3}$$

5. Show that physically we must have
$$1 - n^\top C > 0. \tag{4}$$

6. Given $x \in \mathbb{R}^3$ not at infinity, we write $\hat{x}$ the vector proportional with last coordinate equal to 1. Show if $\{x_i\}$, $i = 1 \dots 4$ are coplanar points in general position, then
$$\begin{pmatrix}\hat{x}_1 & \hat{x}_2 & \hat{x}_3\end{pmatrix}^{-1} \hat{x}_4 \tag{5}$$
makes sense and represents the 3-vector of barycentric coordinates of $\hat{x}_4$ with respect to the other three points.

7. Noting X_L the 3×3 matrix above and X_R the one with corresponding points x', show that
$$H = X_R\, D\, X_L^{-1}, \tag{6}$$
with D a diagonal matrix with positive coefficients.

8. Show that H from (3) has determinant
$$\det H = 1 - n^\top C. \tag{7}$$

9. If $A = U\Sigma V^\top$ the SVD of A, find the SVD of $kA = U_k \Sigma_k V_k^\top$ (distinguish depending on the sign of k).

10. Deduce that using (6) gives exactly H from (3) (without scaling factor), whereas, in view of the results of next section, the regular estimation of H would need some additional checks.

1.2 SVD of H

11. Write an expanded expression of $H^\top H$ from (3).

12. Show that 1 is a singular value of H, give a corresponding singular vector. (We call singular vector a column of the matrix V of the SVD)

13. Show that if $n = \lambda C$, 1 is twice a singular value; what is the remaining singular value? We exclude this case from the following.

14. Show that there is no loss of generality in assuming $C = e_1$ and $n_2 = ae_1 + be_2$ with $n_2 = (a, b) \in \mathbb{R}^2$. Hence, we will be interested in the singular values of the 2×2 matrix $H_2 = I - e_1 \begin{pmatrix} a & b \end{pmatrix}$ (here (e_1, e_2) the canonical basis of $\mathbb{R}^2$).

15. Compute $H_2^\top H_2 e_1$ and $H_2^\top H_2 n_2$ decomposed on the basis (e_1, n_2) (remember that we handled separately the case $n_2 = \lambda e_1$). You can use $\alpha := 1 - a > 0$.

16. Show that any singular vector of H_2 can be written $v = k\, e_1 + n_2$.

17. Eliminating k in the expression of $H_2^\top H_2 v = \lambda v$, show that λ is a root of a degree-2 polynomial:

$$\lambda^2 - (2\alpha + a^2 + b^2)\lambda + \alpha^2 = 0. \tag{8}$$

18. Write $x = a^2 + b^2$ and $y = 2a$, hence $y^2 < 4x$. Show that the eigenvalues of $H_2^\top H_2$ satisfy

$$2(\lambda_\pm - 1) = x - y \pm \sqrt{x}\sqrt{x - 2y + 4}. \tag{9}$$

19. Show that $\lambda_- < 1$.

20. Show that $\lambda_+ > 1$ (fix y, compute the value for $4x = y^2$ and study the variation wrt x).

1.3 Recovering Structure and Motion

21. Write $e_r = RC$. What does e_r represent geometrically?

22. Noting $H = UDV^\top$ with $D = \text{diag}(\sqrt{\lambda_+}, 1, \sqrt{\lambda_-})$ the SVD of H, show that

$$[e_r]_\times UD^2U^\top [e_r]_\times = [e_r]_\times^2. \tag{10}$$

23. Show that

$$[U^\top e_r]_\times (D^2 - I)[U^\top e_r]_\times = 0. \tag{11}$$

24. Show that, up to scale, noting $m_+ = \sqrt{\lambda_+ - 1}$ and $m_- = \sqrt{1 - \lambda_-}$,

$$e_r = U \begin{pmatrix} \pm m_+ \\ 0 \\ m_- \end{pmatrix}. \tag{12}$$

25. Show that

$$[U^\top e_r]_\times (U^\top RV - D) = 0. \tag{13}$$

26. Show that we can write for some $\theta \in \mathbb{R}$

$$U^\top RV = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{14}$$

27. Show that θ must satisfy one system of equations

$$m_- \cos\theta \pm m_+ \sin\theta = \sqrt{\lambda_+}\, m_- \tag{15}$$

$$\mp m_+ \cos\theta + m_- \sin\theta = \mp\sqrt{\lambda_-}\, m_+ \tag{16}$$

28. Show that

$$\cos\theta = \frac{1+\sqrt{\lambda_+\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} := c \tag{17}$$

29. Express the two solutions for R as a function of c, U and V.

30. Plugging the expressions of R into (3), show that

$$Cn^\top = V\,A_1\,V^\top \text{ or } Cn^\top = V\,A_{-1}\,V^\top \tag{18}$$

with

$$A_s = \frac{1}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} \begin{pmatrix} -\sqrt{\lambda_-}m_+^2 & 0 & s\sqrt{\lambda_-}m_-m_+ \\ 0 & 0 & 0 \\ -s\sqrt{\lambda_+}m_-m_+ & 0 & -\sqrt{\lambda_+}m_-^2 \end{pmatrix}. \tag{19}$$

31. Find four vectors $v_1 \dots v_4 \in \mathbb{R}^3$ such that $A_1 = v_1\,v_2^\top$ and $A_{-1} = v_3\,v_4^\top$.

32. Show that we have only two (families of) pairs of solutions: $(C,n) = (k\,Vv_1, \frac{1}{k}Vv_2)$ and $(C,n) = (k\,Vv_3, \frac{1}{k}Vv_4)$ with $k \neq 0$.

33. Recapitulate the former results to propose an algorithm recovering structure and motion (R,C,n) (up to some ambiguity) from the projections of coplanar points in two views.