Maximum likelihood estimation

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Outline

Statistical concepts

Statistical Model

Parametric model – Definition:

Set of distributions parametrized by a vector $\theta \in \Theta \subset \mathbb{R}^p$

$$\mathcal{P}_\Theta = \left\{ p_\theta(x) \mid \theta \in \Theta \right\}$$

Bernoulli model: $X \sim \mathrm{Ber}(\theta) \qquad \Theta = [0, 1]$

$$p_\theta(x) = \theta^x (1-\theta)^{(1-x)}$$

Binomial model: $X \sim \mathrm{Bin}(n, \theta) \qquad \Theta = [0, 1]$

$$p_\theta(x) = \binom{n}{x} \theta^x (1-\theta)^{(1-x)}$$

Multinomial model: $X \sim \mathcal{M}(n, \pi_1, \pi_2, \ldots, \pi_K) \qquad \Theta = [0, 1]^K$

$$p_\theta(x) = \binom{n}{x_1, \ldots, x_k} {\pi_1}^{x_1} \; \ldots \; {\pi_k}^{x_k}$$

Indicator variable coding for multinomial variables

Let C a r.v. taking values in $\{1, \dots, K\}$, with

$$\mathbb{P}(C = k) = \pi_k.$$

We will code C with a r.v. $Y = (Y_1, \dots, Y_K)^\top$ with

$$\boxed{Y_k = 1_{\{C=k\}}}$$

For example if $K = 5$ and $c = 4$ then $\boldsymbol{y} = (0, 0, 0, 1, 0)^\top$.
So $\boldsymbol{y} \in \{0, 1\}^K$ with $\sum_{k=1}^K y_k = 1$.

$$\mathbb{P}(C = k) = \mathbb{P}(Y_k = 1) \quad \text{and} \quad \mathbb{P}(Y = y) = \prod_{k=1}^K \pi_k^{y_k}.$$

Bernoulli, Binomial, Multinomial

$Y \sim \text{Ber}(\pi)$	$(Y_1, \ldots, Y_K) \sim \mathcal{M}(1, \pi_1, \ldots, \pi_K)$
$p(y) = \pi^y (1-\pi)^{1-y}$	$p(\boldsymbol{y}) = \pi_1^{y_1} \ldots \pi_K^{y_K}$

$N_1 \sim \text{Bin}(n, \pi)$	$(N_1, \ldots, N_K) \sim \mathcal{M}(n, \pi_1, \ldots, \pi_K)$
$p(n_1) = \binom{n}{n_1} \pi^{n_1} (1-\pi)^{n-n_1}$	$p(\mathbf{n}) = \begin{pmatrix} & n & \\ n_1 & \ldots & n_K \end{pmatrix} \pi_1^{n_1} \ldots \pi_K^{n_K}$

with

$$\binom{n}{i} = \frac{n!}{(n-i)!i!} \qquad \text{and} \qquad \begin{pmatrix} & n & \\ n_1 & \ldots & n_K \end{pmatrix} = \frac{n!}{n_1! \ldots n_K!}$$

Gaussian model

Scalar Gaussian model : $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$

X real valued r.v., and $\theta = \left(\mu, \sigma^2\right) \in \Theta = \mathbb{R} \times \mathbb{R}_+^*$.

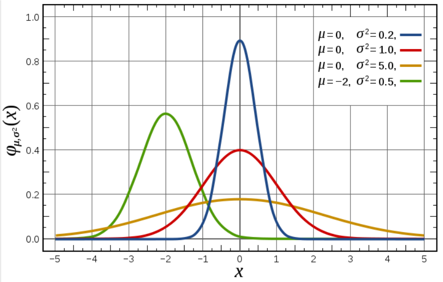
$$p_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

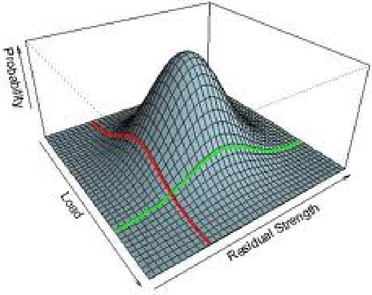
Multivariate Gaussian model: $X \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$

X r.v. taking values in $\mathbb{R}^d$. If $\mathcal{K}_d$ is the set of positive definite matrices of size $d \times d$, and $\theta = \left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right) \in \Theta = \mathbb{R}^d \times \mathcal{K}_d$.

$$p_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^d \det \boldsymbol{\Sigma}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

Gaussian densities





A short review of convex analysis and optimization

Review: convex analysis

Convex function

$$\forall \lambda \in [0,1], \qquad f(\lambda\,\mathbf{x} + (1-\lambda)\,\boldsymbol{y}) \leq \lambda\, f(\mathbf{x}) + (1-\lambda)\, f(\boldsymbol{y})$$

Strictly convex function

$$\forall \lambda \in \,]0,1[, \qquad f(\lambda\,\mathbf{x} + (1-\lambda)\,\boldsymbol{y}) < \lambda\, f(\mathbf{x}) + (1-\lambda)\, f(\boldsymbol{y})$$

Strongly convex function

$$\exists \mu > 0, \text{ s.t.} \quad \mathbf{x} \mapsto f(\mathbf{x}) - \mu \|\mathbf{x}\|^2 \quad \text{is convex}$$

Equivalently:

$$\forall \lambda \in [0,1], \quad f(\lambda\,\mathbf{x} + (1-\lambda)\,\boldsymbol{y}) \leq \lambda\, f(\mathbf{x}) + (1-\lambda)\, f(\boldsymbol{y}) - \mu\,\lambda(1-\lambda)\|\mathbf{x}-\boldsymbol{y}\|^2$$

The largest possible μ is called the strong convexity constant.

Minima of convex functions

Proposition (Supporting hyperplane)

If f is convex and differentiable at $\mathbf{x}$ then

$$f(\boldsymbol{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\boldsymbol{y} - \mathbf{x})$$

Convex function
All local minima are global minima.

Strictly convex function
If there is a local minimum, then it is unique and global.

Strongly convex function
There exists a unique local minimum which is also global.

Minima and stationary points of differentiable functions

Definition (Stationary point)

For f differentiable, we say that $\mathbf{x}$ is a stationary point if $\nabla f(\mathbf{x}) = 0$.

Theorem (Fermat)

If f is differentiable at $\mathbf{x}$ and $\mathbf{x}$ is a local minimum, then $\mathbf{x}$ is stationary.

Theorem (Stationary point of a convex differentiable function)

If f is convex and differentiable at $\mathbf{x}$ and $\mathbf{x}$ is stationary then $\mathbf{x}$ is a minimum.

Theorem (Stationary points of a twice differentiable functions)

For f twice differentiable at $\mathbf{x}$

- *if $\mathbf{x}$ is a local minimum then $\nabla f(\mathbf{x}) = 0$ and $\nabla^2 f(\mathbf{x}) \succeq 0$.*
- *conversely if $\nabla f(\mathbf{x}) = 0$ and $\nabla^2 f(\mathbf{x}) \succ 0$ then $\mathbf{x}$ is a strict local minimum.*

Sample/Training set

The data used to learn or estimate a model typically consists of a collection of observation which can be thought of as instantiations of random variables.

$$X^{(1)}, \dots, X^{(n)}$$

A common assumption is that the variables are **i.i.d.**

- **independent**
- **identically distributed**, i.e. have the same distribution P.

This collection of observations is called

- the *sample* or the *observations* in statistics
- the *sample**s*** in engineering
- the *training set* in machine learning

The maximum likelihood principle

Maximum likelihood principle

- Let $\mathcal{P}_\Theta = \{p(x;\theta) \mid \theta \in \Theta\}$ be a *model*
- Let x be an observation

Sir Ronald Fisher
(1890-1962)

Likelihood:

$$\begin{aligned} \mathcal{L} : \Theta &\to \mathbb{R}_+ \\ \theta &\mapsto p(x;\theta) \end{aligned}$$

Maximum likelihood estimator:

$$\hat{\theta}_{\mathrm{ML}} = \underset{\theta \in \Theta}{\operatorname{argmax}}\, p(x;\theta)$$

Case of i.i.d data

If $(x_i)_{1 \le i \le n}$ is an i.i.d. sample of size n:

$$\hat{\theta}_{\mathrm{ML}} = \underset{\theta \in \Theta}{\operatorname{argmax}} \prod_{i=1}^{n} p_\theta(x_i) = \underset{\theta \in \Theta}{\operatorname{argmax}} \sum_{i=1}^{n} \log\ p_\theta(x_i)$$

The maximum likelihood estimator

The MLE

- does not always exists
- is not necessarily unique
- is not *admissible* in general

MLE for the Bernoulli model

Let $X_1, X_2, \ldots, X_n$ an i.i.d. sample $\sim \text{Ber}(\theta)$. The log-likelihood is

$$\ell(\theta) = \sum_{i=1}^{n} \log p(x_i;\, \theta) = \sum_{i=1}^{n} \log \left[\theta^{x_i}(1-\theta)^{1-x_i}\right]$$

$$= \sum_{i=1}^{n} \big(x_i \log\theta + (1-x_i)\log(1-\theta)\big) = N\log(\theta) + (n-N)\log(1-\theta)$$

with $N := \sum_{i=1}^{n} x_i$.

- $\theta \mapsto \ell(\theta)$ is strongly concave $\Rightarrow$ the MLE exists and is unique.
- since ℓ differentiable + strongly concave its maximizer is the unique stationary point

$$\nabla\ell(\theta) = \frac{\partial}{\partial\theta}\ell(\theta) = \frac{N}{\theta} - \frac{n-N}{1-\theta}.$$

Thus

$$\hat{\theta}_{\text{MLE}} = \frac{N}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

MLE for the multinomial

Done on the board. See lecture notes.

Brief review of Lagrange duality

Convex optimization problem with linear constraints

For

- f a convex function,
- $\mathcal{X} \subset \mathbb{R}^p$ a convex set included in the domain of f,
- $\mathbf{A} \in \mathbb{R}^{n \times p}, \quad \mathbf{b} \in \mathbb{R}^n$,

$$\min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \tag{P}$$

Lagrangian

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T(\mathbf{A}\mathbf{x} - \mathbf{b})$$

with $\boldsymbol{\lambda} \in \mathbb{R}^n$ the *Lagrange multiplier*.

Properties of the Lagrangian

Link between primal and Lagrangian

$$\max_{\boldsymbol{\lambda} \in \mathbb{R}^n} L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{Ax} = \mathbf{b} \\ +\infty & \text{otherwise.} \end{cases}$$

So that

$$\min_{\mathbf{x} \in \mathcal{X}:\ \mathbf{Ax}=\mathbf{b}} f(\mathbf{x}) = \min_{\mathbf{x} \in \mathcal{X}} \max_{\boldsymbol{\lambda} \in \mathbb{R}^n} L(\mathbf{x}, \boldsymbol{\lambda})$$

Lagrangian dual objective function

$$g(\boldsymbol{\lambda}) = \min_{\mathbf{x} \in \mathcal{X}} L(\mathbf{x}, \boldsymbol{\lambda})$$

Dual optimization problem

$$\max_{\boldsymbol{\lambda} \in \mathbb{R}^n} g(\boldsymbol{\lambda}) = \max_{\boldsymbol{\lambda} \in \mathbb{R}^n} \min_{\mathbf{x} \in \mathcal{X}} L(\mathbf{x}, \boldsymbol{\lambda}) \qquad (D)$$

Maxmin-minmax inequality, weak and strong duality

For any $f : \mathbb{R}^n \times \mathbb{R}^m$ and any $w \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, we have

$$\boxed{\max_{z \in Z} \min_{w \in W} f(w, z) \leq \min_{w \in W} \max_{z \in Z} f(w, z).}$$

Weak duality

$$d^* := \max_{\boldsymbol{\lambda} \in \mathbb{R}^n} g(\boldsymbol{\lambda}) = \max_{\boldsymbol{\lambda}} \min_{\mathbf{x} \in \mathcal{X}} L(\mathbf{x}, \boldsymbol{\lambda}) \leq \min_{\mathbf{xx} \in \mathcal{X}} \max_{\boldsymbol{\lambda} \in \mathbb{R}^n} L(\mathbf{x}, \boldsymbol{\lambda}) = \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) =: p^*$$

So that in general, we have $d^* \leq p^*$. This is called weak duality

Strong duality

In some cases, we have strong duality:

- $d^* = p^*$
- Solutions to (P) and (D) are the same

Slater's qualification condition

Slater's qualification condition is a condition on the constraints of a convex optimization problem that guarantees that strong duality holds.

For linear constraints, Slater's condition is very simple:

Slater's condition for a cvx opt. pb with lin. constraints

If there exists an $\mathbf{x}$ *in the relative interior of* $\mathcal{X} \cap \{\mathbf{Ax} = \mathbf{b}\}$ *then strong duality holds.*

MLE for the univariate and multivariate Gaussian

Done on the board. See lecture notes.